THE LAWS OF SOME FREE NILPOTENT GROUPS OF SMALL RANK

by

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CONTENTS

Preface.

This thesis was prepared by me as a post-graduate student in the Department of Mathematics, Institute of Advanced Studies, Australian National University. I am grateful to the same for granting me a Post-graduate Research Scholarship. The early stages of it were spent on entirely different work. I thank Dr R.E. Edwards for his friendly and helpful supervision during that time.

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CHAPTER 0

(0.1) Introduction.

We take for granted all the basic terminology currently in use in the theory of varieties of groups. Most of the notation follows that of the book by Hanna Neumann [3], to which reference is made also whenever standard results are used without proof.

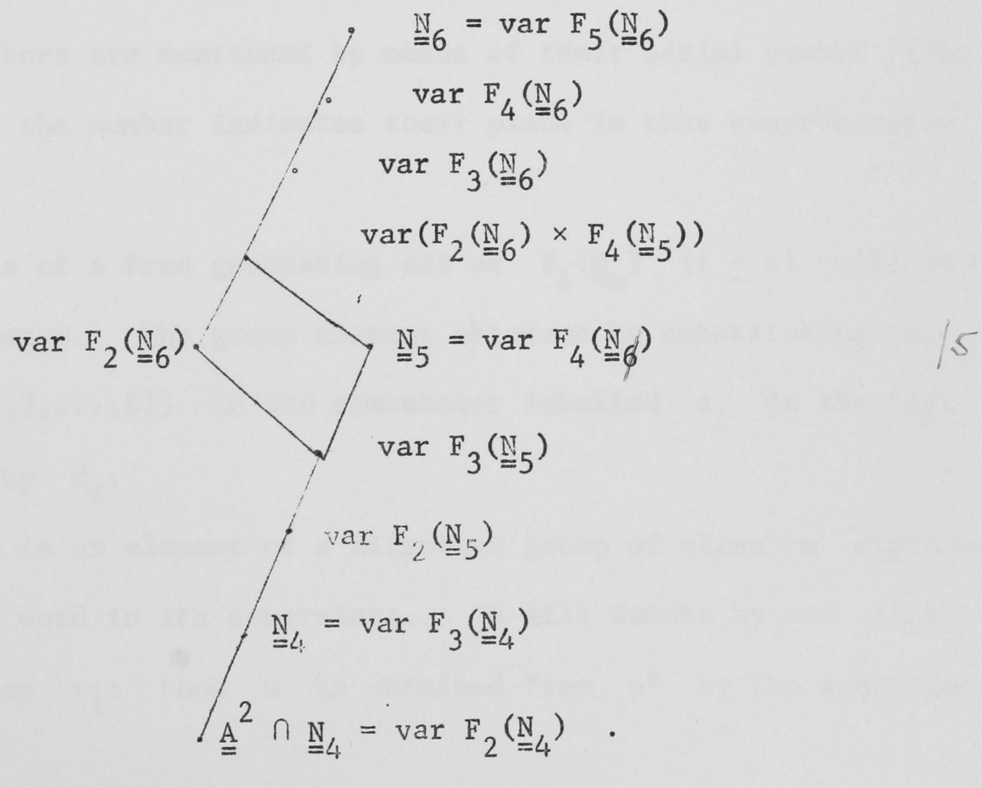
It is known ([3], p.100) that every nilpotent variety of class m is generated by its free group of rank m. This applies in particular to the variety $\underline{\underline{N}}_m$ of all nilpotent groups of class at most m. The question arises which free groups of rank less than m still generate the variety.

A conjecture on this is contained in [3], but this conjecture and some of the supporting evidence offered there have meanwhile been proved false, independently by L.G. Kovács, M.F. Newman, P.F. Pentony [1] and Frank Levin [2]. They prove that if m is an integer greater than 2, then the variety $\underline{\underline{N}}_m$ of all nilpotent groups of class at most m is generated by its free group $F_{m-1}(\underline{\underline{N}}_m)$ of rank $m-1$ but not by its free group $F_{m-2}(\underline{\underline{N}}_m)$ of rank $m-2$. Frank Levin has some more information, namely that the variety generated by the free group $F_{k-1}(\underline{\underline{N}}_m)$ is properly contained in that generated by $F_k(\underline{\underline{N}}_m)$ for $k \leq m-1$.

From these results we see that the free groups $F_k(\underline{\underline{N}}_m)$, $2 \leq k \leq m-2$, do not generate $\underline{\underline{N}}_m$. In general little is known of the varieties generated by them. However, we do know that their laws have finite basis ([3], p.89). That is, all laws are consequences of a finite number of laws amongst them.

The purpose of the present thesis is to determine the varieties of some free groups of small rank, namely, $F_2(\underline{\underline{N}}_5)$, $F_3(\underline{\underline{N}}_5)$, $F_2(\underline{\underline{N}}_6)$, $F_3(\underline{\underline{N}}_6)$, $F_4(\underline{\underline{N}}_6)$, or equivalently, to determine a basis for the laws in these groups. This has been accomplished in all the cases mentioned. It turns out that for each of the free groups $F_2(\underline{\underline{N}}_5)$, $F_3(\underline{\underline{N}}_5)$, $F_3(\underline{\underline{N}}_6)$, $F_4(\underline{\underline{N}}_6)$, a basis consists of laws which are products of commutators of maximal weight, that is of weight five and six respectively. In $F_2(\underline{\underline{N}}_6)$, however, a basis includes a law that is of weight five.

The following diagram depicting the lattice formed by these varieties is then almost immediate.



(0.2) Notation and terminology.

We use the notation $\underline{\underline{U}}$, $\underline{\underline{V}}$, etc., possibly with subscripts, for varieties other than $\underline{\underline{N}}_m$, correspondingly U, V etc. for sets of words that define these varieties, and small roman letters for words, that is elements of a free group X freely generated by the set $x_1,\ldots,x_k$; only $k \leq m$ will be needed. We adopt a definite order for the basic commutators (for definition, see (1.2.1), p.9) in these letters which is such that $x_1 < x_2 < \ldots < x_m$ and generally $c < c'$ certainly when the weight of c is less than that of c'. We do not adopt a definite ordering of all basic commutators of fixed weight, because in any one computation only a subset of the set of all basic commutators will be relevant. All basic commutators that occur in some computation or other are listed in Appendix 1 in the order which we have adopted. It records commutators of the same type together (for example, basic commutators before all other types). Whenever commutators are mentioned by means of their serial number (for example c_{117}), the number indicates their place in that comprehensive list.

The elements of a free generating set of $F_\ell(\underline{\underline{N}}_m)$ $(\ell \leq m)$ will be $g_1,g_2,\ldots,g_\ell$ always. The group element obtained by substituting g_j for x_j $(j \in \{1,2,\ldots,\ell\})$ in the commutator labelled c_i in the list will be denoted by d_i.

Suppose u is an element of a nilpotent group of class m expressed in some way as a word in its generators. We will denote by u^* this word, in variables x_i; thus u is obtained from u^* by the substitution $x_i \to g_i$.

Because practice varies in the literature we record that x^y denotes $y^{-1}xy$ and $[x,y] = x^{-1}y^{-1}xy$.

Where in computations explicit commutators d_i occur, we actually use an abbreviated notation replacing g_j by j. For example, the element $[g_4,g_1,g_2,g_3,g_3]$ will be written $[4,1,2,3,3]$. For commutators that are not left-normed, we use the usual convention; for example $[[x_4,x_3],[x_2,x_1]]$ is written $[x_4,x_3;x_2,x_1]$ leading to the abbreviation $[4,3;2,1]$ for $[[g_4,g_3],[g_2,g_1]]$.

Most computations involve commutators that commute with each other; here additive notation is convenient. In some general statements and in some computations commutators of low weight are involved. Even though they do not commute with each other, we use additive notation for uniformity. Any commutators of weight higher than the class are of course omitted without comment.

The following well known commutator identities will be used without special reference.

$$[x,y+z] = [x,z] + [x,y] + [x,y,z],$$

$$[x+y,z] = [x,z] + [x,z,y] + [y,z],$$

$$[x,y] = -[y,x],$$

$$[x,-y] = -(y+[x,y]-y) = y - [x,y] - y.$$

Repeated application of these identities shows that in a group of class m,

$$[n_1x_1,n_2x_2,\dots,n_mx_m] = n_1n_2 \dots n_m[x_1,x_2,\dots,x_m]$$

for arbitrary integers $n_1,n_2,\ldots,n_m$; in general, if $c(x_1,x_2,\ldots,x_m)$ is a commutator of weight m involving the variables $x_1,x_2,\ldots,x_m$, then

$$c(n_1x_1,n_2x_2,\ldots,n_mx_m) = n_1n_2 \ldots n_m \; c(x_1,x_2,\ldots,x_m).$$

The frequent computations expressing arbitrary commutators in terms of basic commutators are made less tedious by systematic use of the Jacobi-Witt identity; if x,y,z are of weight ℓ,m,n respectively, then

$$[x,y,z] + [y,z,x] + [z,x,y] \equiv 0 \mod X_{(\ell+m+n+1)}$$

The main part of the thesis is contained in chapters 1 - 4. Details of computations are given in the appendices. There (Appendix 1) also we give a table of the relevant basic commutators; some of them do not actually occur in our computations but for simplicity we have included them. Thus, for example, not all of the basic commutators of weight five in the variables x_1,x_2,x_3 occur in our computations but for simplicity of listing we have included all of them.

CHAPTER 1

(1.1) A Summary of Properties of Relatively Free Groups.

The groups $F_k(\underline{\underline{N}}_m)$, that is the free groups of varying rank of the variety $\underline{\underline{N}}_m$, are examples of relatively free groups.

(1.1.1) A relatively free group is a group which is isomorphic to a quotient group $F/U(F)$, where F is free and $U(F)$ is a fully-invariant subgroup of F, that is a verbal subgroup of F.

A number of equivalent defining properties of such groups are basic to our arguments. We list them here.

(1.1.2) ([3], p.9) A relatively free group G possesses a set of generators with the property that every relator of these generators is a law in G.

Such a set of generators is a set of free generators which is perhaps more naturally defined by the following equivalent characterization of a relatively free group.

(1.1.3) ([3], p.9) A relatively free group G possesses a set of generators such that every mapping of these generators into G can be extended to an endomorphism of G.

That every relatively free group is a free group of some variety is indicated by the characterization (1.1.1). We need this fact in the following form.

(1.1.4) ([3], p.12) If $\underline{\underline{V}}$ is the variety defined by the set V of laws, then a group belongs to $\underline{\underline{V}}$ if and only if it is isomorphic to a factor group of $F/V(F)$ for a free group F of suitable rank.

Now (1.1.3) leads to the further fact which we use all the time.

(1.1.5) If $G = F/V(F)$ is a $\underline{\underline{V}}$-free group, or a free group of the variety $\underline{\underline{V}}$, then every mapping of a set of free generators of G into a group of $\underline{\underline{V}}$ can be extended to a homomorphism.

Now every homomorphism of a relatively free group $F/V(F)$ can be induced by a homomorphism of F. This is a slight generalisation of Corollary 13.24 of [3]. This can be extended a little further to give

(1.1.6) If $H = F/U(F)$ is a homomorphic image of $G = F/V(F)$, then every homomorphism of H into a group A is induced by a homomorphism of G into A.

Proof. Let $\underline{\underline{f}}$ be a free generating set of F, μ be the natural homomorphism $\mu : F \to F/U(F)$ and ν be the natural homomorphism $\nu : F \to F/V(F)$. Then $f\mu$ is a free generating set of $F/U(F)$ and $f\nu$ is a free generating set of $F/V(F)$. Since H is a homomorphic image of G, $H \in \underline{\underline{V}}$ and the mapping which maps $f\nu$ onto $f\mu$ for each $f \in \underline{\underline{f}}$ can be extended to a homomorphism of $F/V(F)$ into $F/U(F)$ and this is, in fact, just the natural epimorphism $F/V(F) \twoheadrightarrow F/U(F)$. Now every homomorphism $\alpha : F/U(F) \to A$ is induced by a map $\alpha^* : F \to A$. α^* induces $\alpha' : F/V(F) \to A$, and this induces the given $\alpha : F/U(F) \to A$.

If the variety $\underline{\underline{V}}$ is generated by the group A, that is it is the least variety containing A, then ([3], p.16), the free groups of $\underline{\underline{V}}$ may be constructed as subgroups of cartesian powers of A. In that situation the projections of the free group into the factors provide a set of homomorphisms into A whose kernels have trivial intersection. This leads to the following criterion.

(1.1.7) A group A of $\underline{\underline{V}}$ generates $\underline{\underline{V}}$ if and only if for every $k \geq 1$, there exists a set of homomorphisms of $F_k(\underline{\underline{V}}) = F_k/V(F_k)$ into A whose kernels have trivial intersection.

In the case of a nilpotent variety $\underline{\underline{V}}$, one knows ([3], p.83) that each $F_k(\underline{\underline{V}})$ is in fact a subgroup of a direct power of a finite number of copies of A when A generates $\underline{\underline{V}}$. Hence the set of homomorphisms in (1.1.7) can always be expected to be finite.

Moreover since a nilpotent variety of class m is generated by its free group of rank m, we have the following corollary.

(1.1.8) <u>Corollary</u>. If $\underline{\underline{V}}$ is nilpotent of class m then the free group $F_k(\underline{\underline{V}})$, $k > 1$, generates $\underline{\underline{V}}$ if and only if there exists a (finite) set of homomorphisms of $F_m(\underline{\underline{V}}) = F_m/V(F_m)$ into $F_k(\underline{\underline{V}})$ whose kernels have trivial intersection.

(1.2) Some Remarks on Free Nilpotent Groups.

We adopt the definition of the basic commutators for the free group X generated by $x_1,\ldots,x_n$ as provided in [3]. We give it in the following form.

(1.2.1) Definition.

(i) The letters $x_1,\ldots,x_n$ are basic commutators of weight one, ordered by setting $x_i < x_j$ if $i < j$.

(ii) Suppose basic commutators of weight less than n have been defined and ordered. Call $c_k = [c_i,c_j]$ a basic commutator of weight n if

(a) c_i and c_j are basic and wt. c_i + wt. $c_j = n$,

(b) $c_i > c_j$, and if $c_i = [c_s,c_t]$, then $c_j \geq c_t$.

The basic commutators of weight n occur after those of weight smaller than n in the order, and they are ordered arbitrarily with respect to each other.

Having defined basic commutators in X we state a very important property about words of X, namely their representation in terms of basic commutators. This is contained in the following.

(1.2.2) If X is the free group with free generators $x_1,\ldots,x_n$, then an arbitrary non-trivial element $f \in X$ has a unique representation,

$$f = c_1^{m_1} \ldots c_t^{m_t} \bmod X_{(m+1)}$$

where $c_1,\ldots,c_t$ are basic commutators of weight at most m listed in increasing order and $m_1,\ldots,m_t$ are non-zero integers.

From this one obtains the following two important corollaries.

(1.2.3) Corollary. The basic commutators of weight n form a basis for the free Abelian group $X_{(n)}/X_{(n+1)}$.

(1.2.4) Corollary. The free nilpotent groups $F_k(\underline{\underline{N}}_m)$ are torsion-free.

We use this latter fact again and again in the following way.

(1.2.5) Corollary. If nw is a law in $F_k(\underline{\underline{N}}_m)$ then w is.

As remarked in the introduction it is known that, for $m \geq 3$,

$$\text{var } F_1(\underline{\underline{N}}_m) < \text{var } F_2(\underline{\underline{N}}_m) < \ldots < \text{var } F_{m-2}(\underline{\underline{N}}_m) < \text{var } F_{m-1}(\underline{\underline{N}}_m) = \text{var } F_m(\underline{\underline{N}}_m) = \underline{\underline{N}}_m.$$

We establish a slightly more precise form of (1.1.7) in this particular situation.

(1.2.6) Theorem. Let $W^m_{\ell,k} = \cap \ker \varphi$, where $k < \ell$ and φ runs through all homomorphisms $\varphi : F_\ell(\underline{\underline{N}}_m) \to F_k(\underline{\underline{N}}_m)$. Then $W^m_{\ell,k}$ is fully invariant in $F_\ell(\underline{\underline{N}}_m)$ and $F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k} \cong F_\ell(\text{var } F_k(\underline{\underline{N}}_m))$.

(1.2.7) Corollary. The words $w^* \in X$ corresponding to elements of $W^m_{\ell,k}$ written in some way as words in the free generators of $F_\ell(\underline{\underline{N}}_m)$ are precisely the ℓ-variable laws that distinguish $F_k(\underline{\underline{N}}_m)$ from $F_\ell(\underline{\underline{N}}_m)$ (that is, that hold in the one but not in the other).

Proof of (1.2.6) Let $w \in W^m_{\ell,k}$, ψ an endomorphism of $F_\ell(\underline{\underline{N}}_m)$ and α a homomorphism of $F_\ell(\underline{\underline{N}}_m)$ into $F_k(\underline{\underline{N}}_m)$. Then $\psi\alpha$ is a homomorphism of

$F_\ell(\underline{\underline{N}}_m)$ into $F_k(\underline{\underline{N}}_m)$ and $w(\psi\alpha) = (w\psi)\alpha = 0$. So we have $w\psi \in W^m_{\ell,k}$. Hence $W^m_{\ell,k}$ is fully invariant in $F_\ell(\underline{\underline{N}}_m)$. Therefore, by (1.1.1), $F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k}$ is relatively free. Put $G = F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k}$; and let $g \in \ker\psi$, where $\psi : G \to F_k(\underline{\underline{N}}_m)$. Let $\bar{g}$ be an element in $F_\ell(\underline{\underline{N}}_m)$ such that $g = \bar{g}\, W^m_{\ell,k}$. By (1.1.6) ψ is induced by a homomorphism $\varphi : F_\ell(\underline{\underline{N}}_m) \to F_k(\underline{\underline{N}}_m)$. This means that $\bar{g}\varphi = g\psi = 0$, and so $\bar{g} \in W^m_{\ell,k}$, that is g is trivial in G. Thus G is isomorphic to a subgroup of a cartesian power of $F_k(\underline{\underline{N}}_m)$ and so $G \in \operatorname{var} F_k(\underline{\underline{N}}_m)$. As G is generated by ℓ elements, it is a homomorphic image of $F_\ell(\operatorname{var} F_k(\underline{\underline{N}}_m))$. Now the laws of $F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k}$ are $[x_1,\ldots,x_{m+1}]$ and those words corresponding to elements in $W^m_{\ell,k}$; all these are satisfied in $F_k(\underline{\underline{N}}_m)$ and hence also in $F_\ell(\operatorname{var} F_k(\underline{\underline{N}}_m))$. Therefore $F_\ell(\operatorname{var} F_k(\underline{\underline{N}}_m)) \in \operatorname{var} F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k}$ and so $F_\ell(\operatorname{var} F_k(\underline{\underline{N}}_m))$ is a homomorphic image of $F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k}$. Hence we conclude that $F_\ell(\underline{\underline{N}}_m)/W^m_{\ell,k} \cong F_\ell(\operatorname{var} F_k(\underline{\underline{N}}_m))$.

Our task is therefore to determine a basis for $W^m_{\ell,k}$, that is a set of words such that $W^m_{\ell,k}$ is the fully invariant closure of it, for the relevant values of k, ℓ, m. Before turning to this task, we record one further easy lemma because it will be used again and again later on.

(1.2.8) Lemma. Suppose U is any set of laws of $F_k(\underline{\underline{N}}_m)$ including the nilpotency law of class m. Then, if $\underline{\underline{U}}$ is the variety defined by U, $F_k(\underline{\underline{N}}_m) = F_k(\underline{\underline{U}})$. Furthermore, U is a basis for the set of all laws in $F_k(\underline{\underline{N}}_m)$ if and only if $F_m(\underline{\underline{U}}) \in \operatorname{var} F_k(\underline{\underline{N}}_m)$.

Proof. Since by assumption the laws in U are laws in $F_k(\underline{\underline{N}}_m)$, $F_k(\underline{\underline{N}}_m) \varepsilon \underline{\underline{U}}$ and so $U(F_k) \subseteq N_m(F_k)$. Similarly $F_k(\underline{\underline{U}}) \varepsilon \underline{\underline{N}}_m$ and $N_m(F_k) \subseteq U(F_k)$. Therefore $N_m(F_k) = U(F_k)$ and we have $F_k(\underline{\underline{N}}_m) = F_k(\underline{\underline{U}})$.

The set U is a basis for the set of all laws of $F_k(\underline{\underline{N}}_m) = F_k(\underline{\underline{U}})$ if and only if $F_k(\underline{\underline{N}}_m)$ generates $\underline{\underline{U}}$; but $\underline{\underline{U}}$ is nilpotent of class at most m, hence it is generated by $F_m(\underline{\underline{U}})$, and so $F_k(\underline{\underline{N}}_m) = F_k(\underline{\underline{U}})$ generates $\underline{\underline{U}}$ if and only if it generates $F_m(\underline{\underline{U}})$.

We now come to a sequence of lemmas which simplify, step by step, the task of finding a basis for $W^m_{\ell,k}$, that is a basis for those laws which hold in $F_k(\underline{\underline{N}}_m)$ but not in $F_\ell(\underline{\underline{N}}_m)$.

(1.2.9) <u>Lemma</u>. For $m \geq 4$, $W^m_{m,m-2} \subseteq F_m(\underline{\underline{N}}_m)_{(m)}$, that is the laws that distinguish $F_{m-2}(\underline{\underline{N}}_m)$ from $\underline{\underline{N}}_m$ are sums of commutators of the full weight m.

Proof. Let w be a law in $F_{m-2}(\underline{\underline{N}}_m)$ which is not a law in $\underline{\underline{N}}_m$. Then w is a commutator word of weight less than or equal to m. Write $w = w_1 + w_2$ where w_2 consists of commutators of weight m only, w_1 consists of commutators of weight less than m. This is possible as commutators of weight m in $F_{m-2}(\underline{\underline{N}}_m)$ commute with an arbitrary element in $F_{m-2}(\underline{\underline{N}}_m)$. Then reducing modulo the mth term of the lower central series of $F_{m-2}(\underline{\underline{N}}_m)$ shows that w_1 is a law in $F_{m-2}(\underline{\underline{N}}_{m-1})$ and is of weight at most $m-1$. By [1] and [2], $F_{m-2}(\underline{\underline{N}}_{m-1})$ generates $\underline{\underline{N}}_{m-1}$, hence all its laws are of weight at least m. Therefore w_1 is trivial and w is of weight m.

We shall see that all the laws of the groups $F_2(\underline{\underline{N}}_5)$ and $F_3(\underline{\underline{N}}_6)$ also are sums of commutators of weight five and six respectively. But in $F_2(\underline{\underline{N}}_6)$ this is no longer true.

For the next two lemmas we need the following definitions.

(1.2.10) Definitions.

(i) Two commutators c_1, c_2 of the same weight and in the variables $x_1, \ldots, x_k$ are said to have the same repetition pattern in these variables if each x_i appears the same number of times in c_1 as it does in c_2.

(ii) A product w of commutators of equal weight is homogeneous if all commutators in w involve the same variables with the same repetition pattern.

(iii) Suppose w consists of commutators of weight m. Then by the collection process, w can be written in the form $w = w_1 + \cdots + w_\ell \bmod X_{(m+1)}$, where each summand is homogeneous and the repetition pattern of different summands is distinct. We call the summands the homogeneous components of w.

(1.2.11) Lemma. If w is a law consisting of commutators of weight m in the group $F_k(\underline{\underline{N}}_m)$ $(k \leq m-2)$, then w is equivalent to the set of its homogeneous components.

Proof. Let w contain h variables, $h \leq m$. We first concentrate on the variable x_1 and split w into parts where x_1 occurs the same number of times in each term. Then x_1 can be repeated at most $m - h + 1$ times, and so w is of the form $w = u_1 + u_2 \dots + u_p$ where $p = m - h + 1$ and u_i is a sum of basic commutators of weight m each of which contains x_1 i times. Let $\varphi_1, \varphi_2, \dots, \varphi_{p-1}$ be maps defined as follows: φ_j takes x_1 into $(j+1)x_1$ and leaves $x_2, \dots, x_h$ unchanged for each $j \in \{1, \dots, p-1\}$. Apply successively the identity map ι and the maps $\varphi_1, \varphi_2, \dots, \varphi_{p-1}$ to w. We obtain

$$\begin{aligned}
u_1 + u_2 + \dots + u_p &= w\iota = w \\
2u_1 + 2^2u_2 + \dots + 2^pu_p &= w\varphi_1 \\
&\dots\dots\dots\dots \\
&\dots\dots\dots\dots \\
pu_1 + p^2u_2 + \dots + p^pu_p &= w\varphi_{p-1} .
\end{aligned}$$

The coefficient determinant of these equations is given by

$$D = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2^2 & 2^3 & \dots & 2^p \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p & p^2 & p^3 & \dots & p^p \end{vmatrix} .$$

This is a special case of Vandermonde's determinant and is clearly non-zero since $1, 2, \dots, p$ are distinct numbers. Hence by the usual method of elimination for solving linear equations, we obtain $Du_i = v_i$

where v_i is a linear combination of w, $w\varphi_1,\ldots,w\varphi_{p-1}$ with integral coefficients. Now since the maps φ_i can be extended to endomorphisms and the image of a law under an endomorphism is also a law, $w\varphi_i$ is a law in $F_k(\underline{\underline{N}}_m)$ for each $i \in \{1,2,\ldots,p\}$. Hence v_i is a law in $F_k(\underline{\underline{N}}_m)$ for each $i \in \{1,2,\ldots,p\}$. But $F_k(\underline{\underline{N}}_m)$ is torsion-free. Hence u_i is a law in $F_k(\underline{\underline{N}}_m)$ for each $i \in \{1,2,\ldots,p\}$, and so the u_i are consequences of w. As w clearly follows from the set $\{u_i, i \in \{1,..,p\}\}$, w is equivalent to this set. Hence w is equivalent to the set of laws obtained by collecting together those commutators in w with the same number of occurrences of x_1. Therefore in each of them at most $h-1$ variables remain to be considered. Hence induction completes the proof and shows that w is equivalent to the set of its homogeneous components.

(1.2.12) <u>Lemma</u>. Every law w in $F_k(\underline{\underline{N}}_m)$ that is homogeneous of weight m and involves less than m variables is equivalent to a set of laws that are homogeneous and involve the full number m of variables.

Proof. Suppose w is in h variables $x_1,\ldots,x_h$, $h < m$ and suppose that (by renaming $x_1,\ldots,x_h$ if necessary) in w, $x_1,\ldots,x_q$ $(q \leq h)$ are repeated K times where K is maximal. The commutator identities permit us to assume that w is a sum of commutators whose entries are single variables or their inverses.

<u>Step 1</u>. Substitute $x_1 + x_{h+1}$ for x_1 in w. Expand the result using the identities listed in the introduction so as to obtain a sum of

commutators whose entries are again single variables. Let w_1 be the sum of commutators so obtained. Again the identities show that the result will contain all the original commutators that occurred in w, also all these same ones with x_1 replaced by x_{h+1}; therefore the collection process enables us to write w_1 in the form $w_1 = w + v_1 + v_2$ where (a) v_1 can be obtained from w by substituting x_{h+1} for x_1 and (b) $v_2 = v_{21} + v_{22} + \ldots + v_{2(K-1)}$ where v_{2j} consists of commutators in which x_1 occurs j times and x_{h+1} $K-j$ times for $j = 1,2,\ldots,K-1$. Since w_1 is obtained from w by the endomorphism φ which maps $x_1 \to x_1 + x_{h+1}$, $x_r \to x_r$ for each $r \in \{z,\ldots,h\}$, w_1 is a law in $F_k(\underline{\underline{N}}_m)$. Similarly v_1 is a law in $F_k(\underline{\underline{N}}_m)$ as it is obtained from w by renaming one variable; therefore v_2 is a law in $F_k(\underline{\underline{N}}_m)$. Now substitute x_1 for x_{h+1} in v_2. Then v_2 goes into λw where λ is an integer greater than or equal to two. Hence the law v_2 implies λw, and therefore it implies w by (1.2.5). Thus v_2 and w are equivalent. But in v_2 one of the variables $x_1,\ldots,x_q$ is repeated less than K times; the same is therefore true for the homogeneous components of v_2 and so by (1.2.11), w is equivalent to a set of laws in which only $q-1$ variables are repeated K times, and all other variables (including the new variable x_{h+1}) are repeated less than K times. After q steps, w is equivalent to a set of laws in which all variables are repeated less than K times, hence induction completes the proof.

(1.2.13) Lemma. Suppose that w is a non-trivial law in $F_k(\underline{\underline{N}}_m)$, $k \leq m-2$ and that w is written as a sum of basic commutators. Then each of the basic commutators in w involves at least $k+1$ variables.

Proof. Let h be the total number of variables involved. If $h \leq k$, we replace x_i by g_i in w; the result is the trivial element in $F_k(\underline{\underline{N}}_m)$. Hence by (1.2.2), the coefficient of every basic commutator $c(g_1,\ldots,g_h)$ vanishes so that w is trivial, a contradiction to our assumption. So we may now assume that $h > k$, that is some commutators that occur in w must contain more than k variables. We want to show that all do. Let $\{x_{i_1},x_{i_2},\ldots,x_{i_k}\}$ be an arbitrary subset of k elements of $\{x_1,x_2,\ldots,x_h\}$. Let φ be the endomorphism of the free group X_h defined by:

$$x_{i_j} \to x_{i_j} \quad \text{for } j \in \{1,2,\ldots,k\},$$

$$x_r \to 0 \quad \text{for } r \in \{1,2,\ldots,h\} \setminus \{i_1,i_2,\ldots,i_k\} .$$

Now $w\varphi$ consists of all those basic commutators in the variables $x_{i_1},x_{i_2},\ldots,x_{i_k}$ in w. Since $w\varphi$ is again a law in $F_k(\underline{\underline{N}}_m)$, then, if x_{i_j} is replaced by g_j in $w\varphi$, the result is the trivial element in $F_k(\underline{\underline{N}}_m)$. Again by (1.2.2) the coefficients of the corresponding basic commutators in $w\varphi$ vanish so that $w\varphi$ is trivial. Since $x_{i_1},x_{i_2},\ldots,x_{i_k}$ is an arbitrary set of k variables, we conclude that the coefficients of the basic commutators in any set of k variables vanish. Hence the coefficients of all basic commutators in less than or equal to k variables vanish. That is, each of the basic commutators in w involves at least $k+1$ variables.

(1.2.14) <u>Lemma</u>. Suppose a basis for the laws of $F_k(\underline{\underline{N}}_m)$ consists of laws involving only commutators of weight greater than or equal to h where (by (1.2.13)) $h > k$, and, of course, $h \leq m$. Then every non-trivial law w in $F_k(\underline{\underline{N}}_{m+1})$ can be written as a sum of basic commutators of weight greater than or equal to h.

Proof. Since $F_k(\underline{\underline{N}}_m)$ is a homomorphic image of $F_k(\underline{\underline{N}}_{m+1})$, w is a law in $F_k(\underline{\underline{N}}_m)$. Now w follows from laws involving only commutators of weight greater than or equal to h. Hence w itself has the same property.

Finally we record a method to obtain laws that distinguish $F_k(\underline{\underline{N}}_m)$ from $\underline{\underline{N}}_m$ provided that laws are known for $F_k(\underline{\underline{N}}_{m-1})$ which are not laws of $\underline{\underline{N}}_{m-1}$.

(1.2.15) ([3], p.101) Suppose w is a law in $F_k(\underline{\underline{N}}_m)$ but not in $\underline{\underline{N}}_m$. Then $[w,x_{m+1}]$ is a law in $F_k(\underline{\underline{N}}_{m+1})$ but not in $\underline{\underline{N}}_{m+1}$.

CHAPTER 2

THE LAWS OF THE FREE GROUPS $F_2(\underline{\underline{N}}_5)$, $F_3(\underline{\underline{N}}_5)$

(2.1) The laws of $F_3(\underline{\underline{N}}_5)$.

From [1], we obtain in $F_3(\underline{\underline{N}}_5)$ the law $\sum_{\sigma \varepsilon S} \varepsilon(\sigma)[x_5, x_{4\sigma}, x_{3\sigma}, x_{2\sigma}, x_{1\sigma}]$, where S is the group of permutations of $\{1,2,3,4\}$, $\varepsilon(\sigma) = 1$ or -1 according as σ is an even or odd permutation. In terms of basic commutators, (see (A.2.1.1), p.A.11 - A.12), this can be written in the form:

$$w(3,5) = -[x_4, x_1, x_5; x_3, x_2] + [x_4, x_2, x_5; x_3, x_1] - [x_4, x_3, x_5; x_2, x_1]$$
$$+[x_3, x_1, x_5; x_4, x_2] - [x_3, x_2, x_5; x_4, x_1] - [x_2, x_1, x_5; x_4, x_3].$$

We want to show that $w(3,5)$ and the nilpotency law form a basis of laws for $F_3(\underline{\underline{N}}_5)$.

Denote by $\underline{\underline{U}}$ the variety defined by $\{w(3,5), [x_1, x_2, \ldots, x_6]\}$. By lemma (1.2.8) we have to show that $F_5(\underline{\underline{U}})$ is generated by $F_3(\underline{\underline{U}}) = F_3(\underline{\underline{N}}_5)$. The first step consists in using the law $w(3,5)$ to eliminate as far as possible superfluous generators (that is, basic commutators of weight five in the free generators of $F_5(\underline{\underline{U}})$) from the lowest term of $F_5(\underline{\underline{U}})$. This procedure occurs again and again. The law $w(3,5)$ gives a number of relations between basic commutators of weight five in $F_5(\underline{\underline{U}})$, obtained from $w(3,5)$ by substituting generators for the variables in various ways. These relations constitute a set of linear equations between generators of the abelian group $(F_5(\underline{\underline{U}}))_{(5)}$. We use the ordinary method of elimination in solving linear equations. In each step, we use a relation to eliminate a basic commutator, that is to express a basic commutator in the relation in terms of other basic commutators in the relation, and substitute this expression of the basic commutator for the same commutator whenever it occurs in the remaining relations. Thus after each such step the number of unused relations is reduced by 1; furthermore these relations do not involve the basic commutators which have hitherto been eliminated. If there are originally $k \geq 1$ relations, then after $0 \leq h \leq k$ steps, h basic commutators will have been eliminated and at most $k-h$ relations are left over unused. In some cases, for a given set of relations, a set of basic commutators can be eliminated in this way and no relations are left over unused. In other cases, however, some relations are left over unused and we cannot carry on our process of elimination any further. For example, in one of the cases that we consider, a relation of the form $2x + 2y + 2z = 0$ is left over unused. We cannot use this relation to express any one of the three basic commutators x,y,z in terms of the other two, but it indicates that $x + y + z$ may be an element of order 2 in $F_5(\underline{\underline{U}})$.

In the present case, the following basic commutators are eliminated (see (A.2.1.2), p.A.12-A.14). $d_{117} = [2,1,5;4,3]$, $d_{123} = [2,1,4;5,3]$, $d_{125} = [2,1,3;5,4]$, $d_{126} = [3,1,2;5,4]$.

Now by corollary (1.1.8) a sufficient condition for $\underline{\underline{U}}$ to be generated by $F_3(\underline{\underline{U}})$ is that there exists a set of homomorphisms of $F_5(\underline{\underline{U}})$ into

$F_3(\underline{\underline{U}})$ whose kernels intersect trivially. This condition is actually satisfied as the following argument will show.

Let $v \in \overline{W}^5_{5,3}$, where $\overline{W}^5_{5,3} = \{\cap \ker\varphi / \varphi: F_5(\underline{\underline{U}}) \to F_3(\underline{\underline{U}})\}$. Suppose, to the contrary, that $v \neq 0$. Then the corresponding word v^* is a law which distinguishes $F_3(\underline{\underline{N}}_5)$ from $\underline{\underline{N}}_5$. Hence by lemma (1.2.9) v^* (and so v) consists of commutators which are of weight 5. By lemmas (1.2.11) and (1.2.12) v^* is equivalent to a set of laws that are homogeneous and involve 5 variables. Hence v follows from words in $\overline{W}^5_{5,3}$ that are homogeneous and involve precisely 5 generators and we may therefore assume that v is homogeneous and involves precisely 5 generators. Now represent v by a linear combination with integral coefficients of basic commutators of weight 5 in precisely 5 generators, where we may assume that the basic commutators d_{117}, d_{123}, d_{125}, d_{126} are not present in this representation since in $F_5(\underline{\underline{U}})$ they have been eliminated. Map v by the following homomorphisms φ_i, $i = 1,2,\ldots,8$ of $F_5(\underline{\underline{U}})$ into $F_3(\underline{\underline{U}})$.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
$1 \to$	1	1	1	1	1	1	1	1
$2 \to$	2	2	2	1	1	1	2	2
$3 \to$	2	2	3	2	2	1	1	3
$4 \to$	2	3	3	2	3	2	3	2
$5 \to$	3	3	3	3	3	3	2	3 .

Let $\varphi \in \{\varphi_1,\varphi_2,\ldots,\varphi_8\}$. Represent $v\varphi$ by a linear combination with integral coefficients of basic commutators of weight 5 in precisely 3 generators. Now (1.2.2) with $v\varphi = 0$ implies that these coefficients vanish. By equating these coefficients (which involve one or more of the coefficients of the basic commutators in v) to zero, we obtain a set of linear equations in the coefficients of the basic commutators in v. If we let φ run through the set $\{\varphi_1, \varphi_2, \ldots, \varphi_8\}$, we obtain a system of linear equations. We solve this system of linear equations and we find that the only solution is the trivial one. Hence v is trivial and the original supposition that $v \neq 0$ is false (see A.(2.1.3.), p.A. 14-A. 19). Therefore we have:

(2.1.1) <u>Theorem</u>. The set of laws $\{w(3,5), [x_1, x_2, \ldots, x_6]\}$ is a basis of laws for $F_3(\underline{\underline{N}}_5)$.

This now also shows, as indicated earlier, that the basic commutators of weight 5 in 5 generators other than $d_{117}, d_{123}, d_{125}, d_{126}$ are in fact independent in $F_5(\underline{\underline{U}})$. But since we did not need to consider commutators in less than 5 generators, this gives no indication of the total rank of $(F_5(\underline{\underline{U}}))_{(5)}$, and so is of no further interest.

(2.2) The laws of $F_2(\underline{\underline{N}}_5)$.

As in the previous problem we find some laws that hold in $F_2(\underline{\underline{N}}_5)$ but not in $\underline{\underline{N}}_5$ and then show that together with the nilpotency law they form a basis of laws for $F_2(\underline{\underline{N}}_5)$.

Now $F_2(\underline{\underline{N}}_4)$ does not generate $\underline{\underline{N}}_4$ and so a basis of laws for $F_2(\underline{\underline{N}}_4)$ includes laws that hold in $F_2(\underline{\underline{N}}_4)$ but not in $\underline{\underline{N}}_4$. The method described in (1.2.15) enables us to construct from each such law a law that holds in $F_2(\underline{\underline{N}}_5)$ but not in $\underline{\underline{N}}_5$. So accordingly we proceed to find a basis of laws for $F_2(\underline{\underline{N}}_4)$ and then construct laws for $F_2(\underline{\underline{N}}_5)$ from laws in this basis.

From 35.33 ([3], p.102), we see that $w(2,4) = [x_1,x_2;x_3,x_4]$ is a law in $F_2(\underline{\underline{N}}_4)$ but not in $\underline{\underline{N}}_4$. Now Theorem 36.34 ([3], p.107) implies that $\underline{\underline{U}} = \text{var } F_2(U)$ where $\underline{\underline{U}}$ is the variety defined by the set of laws $U = \{w(2,4), [x_1,x_2,\ldots,x_5]\}$. Since $F_2(\underline{\underline{U}}) = F_2(\underline{\underline{N}}_4)$ (by Lemma (1.2.8)), $\underline{\underline{U}} = \text{var } F_2(\underline{\underline{N}}_4)$, that is $\underline{\underline{U}}$ is a basis for the laws of $F_2(\underline{\underline{N}}_4)$. It follows from (1.2.15) that $w_1(2,5) = [x_1;x_2;x_3,x_4;x_5]$ is a law distinguishing $F_2(\underline{\underline{N}}_5)$ from $\underline{\underline{N}}_5$. Expressing $w_1(2,5)$ in terms of basic commutators, we obtain (see (A.2.2.1), p.A.19):

$$w_1(2,5) = [x_2,x_1,x_5;x_4,x_3] - [x_4,x_3,x_5;x_2,x_1].$$

Now $F_2(\underline{\underline{N}}_5)$ being a homomorphic image of $F_3(\underline{\underline{N}}_5)$, $w(3,5)$ (p.19) is a law in $F_2(\underline{\underline{N}}_5)$. Furthermore, if we put

$$w_2(2,5) = [x_2,x_1,x_5;x_4,x_3] + [x_3,x_2,x_5;x_4,x_1] - [x_3,x_1,x_5;x_4,x_2],$$

we note that $w_1(2,5)$, $w(3,5)$ imply that $2w_2(2,5)$ is a law in $F_2(\underline{\underline{N}}_5)$ (see (A.2.2.2), p.A.19) so that (by Corollary (1.2.5)) $w_2(2,5)$ also is a law in $F_2(\underline{\underline{N}}_5)$.

We want to show that $w_1(2,5)$, $w_2(2,5)$ and the nilpotency law form a basis of laws for $F_2(\underline{\underline{N}}_5)$.

Denote by $\underline{\underline{U}}_1$ the variety defined by $U_1 = \{w_1(2,5), w_2(2,5), [x_1,x_2,\ldots,x_6]\}$. By a procedure similar to that described in (2.1) we eliminate the following basic commutators of weight 5 in precisely the 5 generators of $F_5(\underline{\underline{U}}_1)$ (see (A.2.2.3), p.A.20-A.21).

$$d_{107} = [4,3,5;2,1], \quad d_{108} = [5,3,4;2,1], \quad d_{109} = [4,2,5;3,1],$$
$$d_{110} = [5,2,4;3,1], \quad d_{111} = [4,1,5;3,2], \quad d_{112} = [5,1,4;3,2],$$
$$d_{114} = [5,2,3;4,1], \quad d_{116} = [5,1,3;4,2], \quad d_{117} = [2,1,5;4,3],$$
$$d_{118} = [5,1,2;4,3], \quad d_{120} = [4,2,3;5,1], \quad d_{123} = [2,1,4;5,3],$$
$$d_{124} = [4,1,2;5,3], \quad d_{125} = [2,1,3;5,4], \quad d_{126} = [3,1,2;5,4].$$

Again we let $v \in \overline{W}^5_{5,2}$, where $\overline{W}^5_{5,2} = \{\cap \ker \varphi \mid \varphi : F_5(\underline{\underline{U}}_1) \to F_2(\underline{\underline{U}}_1)\}$, and proceed to show that $v = 0$. Assume, to the contrary, that $v \neq 0$. Write v as a linear combination with integral coefficients of basic commutators in the generators of $F_5(\underline{\underline{U}}_1)$. The corresponding word v^* is a non-trivial law of $F_2(\underline{\underline{N}}_5)$. Hence by lemmas (1.2.13) and (1.2.14) each basic commutator in v^* has weight greater than or equal to 4 and involves at least 3 variables. So each basic commutator in v has weight greater than or equal to 4 and involves at least 3 generators. We will first show that all basic commutators in v have weight 5.

Map v by the following homomorphisms of $F_5(\underline{\underline{U}}_1)$ into $F_2(\underline{\underline{U}}_1)$.

		φ_{11}	φ_{12}	φ_{13}
1	$\to$	1	1	[2,1]
2	$\to$	2	[2,1]	1
3	$\to$	[2,1]	2	2
4	$\to$	0	0	0
5	$\to$	0	0	0

We need only consider the images of the basic commutators in v of weight 4 in precisely the generators 1,2,3 since under φ_{1i}, $i \in \{1,2,3\}$, any basic commutator in v which involves the generators 4 or 5 goes into the trivial element, and any basic commutator in v of weight 5 in precisely the generators 1,2,3 go into a commutator of weight at least 6 and hence is trivial. The coefficients of the basic commutators in v of weight 4 in precisely the generators 1,2,3 vanish (see (A.2.2.4), p.A. 22–A. 23).
Similarly, if $i_1, i_2, i_3 \in \{1,2,\ldots,5\}$, $i_1 < i_2 < i_3$, then if we map v by the homomorphisms which take

i_1	$\to$	i_1	i_1	$[i_2,i_1]$
i_2	$\to$	i_2	$[i_2,i_1]$	i_2
i_3	$\to$	$[i_2,i_1]$	i_3	i_3

and the remaining two generators to zero, we see that the coefficients of the basic commutators in v of weight 4 in precisely the generators i_1, i_2, i_3 vanish. It follows therefore that the coefficients of the basic commutators in v of weight 4 in precisely 3 generators vanish. Now map v by the homomorphisms:

		φ_{21}	φ_{22}	φ_{23}	φ_{24}	φ_{25}	φ_{26}
1	$\to$	1	1	1	1	1	1
2	$\to$	[2,1]	1	1	2	2	2
3	$\to$	1	[2,1]	1	[2,1]	1	2
4	$\to$	1	1	[2,1]	2	[2,1]	[2,1]
5	$\to$	0	0	0	0	0	0

We need now only consider the images of the basic commutators in v of weight 4 in precisely the generators $1,2,3,4$. The coefficients of such commutators are seen to vanish (see (A.2.2.5), p.A.23-A.24). Hence we conclude that the coefficients of the basic commutators in v of weight 4 in precisely 4 generators vanish. Thus we have shown that v consists of basic commutators of weight 5. Write v in the form $v = u_1 + u_2 + u_3$ where u_1, u_2, u_3 are the sums of the homogeneous components of v in $3,4$ and 5 generators respectively. By Lemma (1.2.12), u_1^*, u_2^*, u_3^* are laws in $F_2(\underline{\underline{N}}_5) = F_2(\underline{\underline{U}}_1)$ and are consequences of homogeneous laws of weight 5 in 5 variables. Hence u_1, u_2, u_3

are in $\overline{W}^5_{5,2}$ and are consequences of homogeneous words in $\overline{W}^5_{5,2}$ of weight 5 in 5 generators.

We therefore may now assume that v is homogeneous of weight 5 in 5 generators. Because some of the basic commutators of weight 5 in precisely 5 generators have been eliminated (p.24), we see that v is a linear combination with integral coefficients of the following basic commutators.

$[2,1,3,4,5], [3,1,2,4,5], [4,1,2,3,5], [5,1,2,3,4], [3,2,5;4,1], [3,1,5;4,2],$
$[3,2,4;5,1], [3,1,4;5,2], [4,1,3;5,2].$

Map v by the homomorphisms:

		φ_{31}	φ_{32}	φ_{33}	φ_{34}	φ_{35}	φ_{36}	φ_{37}	φ_{38}	φ_{39}
1	$\to$	1	1	1	1	1	1	1	2	1
2	$\to$	2	1	1	1	1	2	1	1	1
3	$\to$	1	2	1	1	1	2	2	2	2
4	$\to$	1	1	2	1	2	2	1	1	2
5	$\to$	1	1	1	2	2	1	2	1	1 .

The coefficients of the above basic commutators are seen to vanish (see (A.2.2.6), p.A. 24-A. 25). Hence v is trivial and the supposition that $v \neq 0$ is false and we have:

(2.2.1) <u>Theorem</u>. The set of laws $\{w_1(2,5), w_2(2,5), [x_1, x_2, \ldots, x_6]\}$ is a basis of laws for $F_2(\underline{\underline{N}}_5)$.

Finally we mention that by a method involving tedious computations we found that in $F_5(\underline{\underline{U}}_1)$, $w_2(2,5)$ is in fact an element of order 2 and that in $F_3(\underline{\underline{U}}_1)$, $F_4(\underline{\underline{U}}_1)$ there are no elements of order 2. This shows that the variety of all centre-extended-by-metabelian nilpotent of class 5 groups is generated by its free group of rank 5 but no free group of lower rank.

CHAPTER 3

THE LAWS OF THE FREE GROUPS $F_3(\underline{\underline{N}}_6)$, $F_4(\underline{\underline{N}}_6)$

(3.0) A remark on basic commutators of weight 6.

It is convenient to have, for use in this chapter and the next, a description of the relation between the four types of basic commutators that occur when such a commutator whose entries are group elements of X is expanded into a sum of basic commutators. We use the following terminology for the commutators other than left-normed. A commutator of the form $[x,y,z,t;u,v]$ is of type $(4,2)$, a commutator of the form $[x,y;z,t;u,v]$ is of type $(2,2,2)$, and a commutator of the form $[x,y,z;t,u,v]$ is of type $(3,3)$.

(3.0.1) Lemma. If the entries in a basic commutator, c say, are permuted, then

(i) if c is left-normed, the expansion of the result in basic commutators may include commutators of each of the other types;

(ii) if c is of type $(4,2)$, the expansion will include commutators of type $(4,2)$ or $(2,2,2)$ only;

(iii) if c is of type $(2,2,2)$, the expansion again consists of commutators of type $(2,2,2)$ only;

(iv) if c is of type $(3,3)$, the expansion again consists of commutators of type $(3,3)$ only.

(3.0.2) <u>Corollary</u>. All statements remain true for the expansion of the commutator obtained from c by substituting arbitrary words.

Proof. (i) We give an example: the expansion of the left-normed commutator $[x_2,x_1,x_4,x_3,x_6,x_5]$ in basic commutators contains commutators of each of the 4 types.

$$[x_2,x_1,x_4,x_3,x_6,x_5] = -[x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_3,x_5;x_2,x_1,x_6]$$
$$-[x_4,x_3,x_5,x_6;x_2,x_1] + [x_2,x_1,x_5,x_6;x_4,x_3] + [x_2,x_1,x_3,x_4;x_6,x_5]$$
$$-[x_4,x_3;x_2,x_1;x_6,x_5] + [x_2,x_1,x_3,x_4,x_5,x_6].$$

The expansion in basic commutators is obtained by the use of the Jacobi identity combined with permutation of the first two entries or the first two entries following a semicolon (the entries may be single variables or commutators of weight greater than 1), as is explained more fully in the remaining parts of the proof.

(ii) If in $[x,y,z,t;u,v]$, where the entries are variables, $u < v$, exchange u and v; then the commutator is replaced by its negative and the last two variables are in the correct order. Similarly, if $x < y$, exchange these; this again merely replaces the commutator by its negative without changing the type. We now assume that $x > y$ and $u > v$. Now, if $[x,y,z]$ is not basic, that is if $y > z$, use Jacobi's identity to write $[x,y,z] = -[y,z,x] - [z,x,y] \bmod X_{(4)}$. Then linearity and permutation of entries gives

$$[x,y,z,t;u,v] = -[y,z,x,t;u,v] - [z,x,y,t;u,v]$$
$$= -[y,z,x,t;u,v] + [x,z,y,t;u,v],$$

that is $[x,y,z,t;u,v]$ is expressed in terms of two commutators of the same type in which the first three entries form a basic commutator. Assume therefore now that in $[x,y,z,t;u,v]$ $u > v$ and $x > y$, $z \geq y$. If now $z > t$, write by means of the Jacobi identity again,

$$[x,y;z,t] = -[z,t;x,y] - [t,[x,y],z] \bmod X_{(5)}$$
$$= -[z,t;x,y] + [x,y,t,z] \bmod X_{(5)}$$

giving

$$[x,y,z,t;u,v] = -[z,t;x,y;u,v] + [x,y,t,z;u,v].$$

If here $[z,t] > [x,y] < [u,v]$, then $[z,t;x,y;u,v]$ is basic. If $[z,t] < [x,y]$, exchange $[z,t]$, $[x,y]$ (here the entries permuted are commutators of weight 2). So we may assume that $[z,t] > [x,y]$. Suppose that $[x,y] > [u,v]$. Then the Jacobi identity and permutation of entries gives at once

$$[z,t;x,y;u,v] = -[x,y;u,v;z,t] - [u,v;z,t;xy]$$
$$-[x,y;u,v;z,t] + [z,t;u,v;x,y]$$

which are basic. Now if $t \geq y$ the commutator $[x,y,t,z;u,v]$ is basic. If $t < y$, again by the Jacobi identity and permutation of entries, $[x,y,t,z;u,v]$ can be expressed as a sum of two basic terms of type (4,2).

(iii) We may assume that $x > y$, $t > u$. If $[x,y,z]$ is not basic write by means of Jacobi's identity and permutation of entries $[x,y,z;t,u,v]$ in terms of commutators of the same type in which the first three entries form a basic commutator and the last three are t,u,v.

If $[t,u,v]$ is not basic, the same process expresses each of these commutators in terms of commutators of type $(3,3)$ in which the first three entries and the last three entries form basic commutators. So assume $[x,y,z]$, $[t,u,v]$ are basic in $[x,y,z;t,u,v]$. If this is not basic, that is if $[x,y,z] < [t,u,v]$, exchange $[x,y,z]$, $[t,u,v]$.

(iv) Assume that $x > y$, $z > t$, $u > v$ and that $[x,y] > [z,t]$. Then if $[x,y;z,t;u,v]$ is not basic, that is if $[z,t] > [u,v]$ obtain $[x,y;z,t;u,v]$ in terms of basic commutators of the same type by using Jacobi's identity and by permuting entries.

Proof of Corollary. This is immediate: from linearity, one first obtains a sum of commutators of the same type as c whose entries are variables; to these the lemma applies.

(3.1) <u>The laws of</u> $F_4(\underline{\underline{N}}_6)$.

The law of $F_4(\underline{\underline{N}}_6)$ that we obtain from [1] in this case is $w(4,6)$, $w(4,6) = \sum_{\sigma\varepsilon S} \mathcal{E}(\sigma)[x_6,x_{5\sigma},x_{4\sigma},x_{3\sigma},x_{2\sigma},x_{1\sigma}]$, where S is the group of permutations of $\{1,2,3,4,5\}$, $\mathcal{E}(\sigma) = 1$ or -1 according as σ is an even or odd permutation. Let K be the subgroup of S generated by the transpositions $(2i-1\ \ 2i)$ with $i \varepsilon \{1,2\}$. Then by [1] again, $w(4,6)$ can be rewritten in the form:

$$w(4,6) = \sum_{\tau\varepsilon T} \varepsilon(\tau)[[x_6,x_{5\tau}],[x_{4\tau},x_{3\tau}],[x_{2\tau},x_{1\tau}]],$$

where T is an arbitrary transversal of K in S. An expression for $w(4,6)$ in terms of basic commutators is given in Appendix 3 (see (A.3.1.1), p.A.26.).

We again want to eliminate as many of the basic commutators as possible that, in $F_6(\underline{\underline{N}}_6)$, become dependent as a consequence of adding the law $w(4,6)$. It is tedious to cpmpute relations from $w(4,6)$ as it involves a large number of terms. We have, however, found two laws in $F_4(\underline{\underline{N}}_6)$ which we derive with the aid of $w(4,6)$ and which involve less terms. They are $w_1(4,6)$ and $w_2(4,6)$ (see (A.3.1.2), p.A.26.) for their explicit forms). To derive these laws, we set

$$\begin{aligned}
x = &-[x_6,x_4;x_3,x_2;x_5,x_1] + [x_6,x_3;x_4,x_2;x_5,x_1] - [x_6,x_2;x_4,x_3;x_5,x_1] \\
&+[x_6,x_4;x_3,x_1;x_5,x_2] - [x_6,x_3;x_4,x_1;x_5,x_2] + [x_6,x_1;x_4,x_3;x_5,x_2] \\
&-[x_6,x_4;x_2,x_1;x_5,x_3] + [x_6,x_2;x_4,x_1;x_5,x_3] - [x_6,x_1;x_4,x_2;x_5,x_3] \\
&+[x_6,x_3;x_2,x_1;x_5,x_4] - [x_6,x_2;x_3,x_1;x_5,x_4] + [x_6,x_1;x_3,x_2;x_5,x_4], \\
y = &-[x_5,x_4;x_3,x_2;x_6,x_1] + [x_5,x_3;x_4,x_2;x_6,x_1] - [x_5,x_2;x_4,x_3;x_6,x_1] \\
&+[x_5,x_4;x_3,x_1;x_6,x_2] - [x_5,x_3;x_4,x_1;x_6,x_2] + [x_5,x_1;x_4,x_3;x_6,x_2] \\
&-[x_5,x_4;x_2,x_1;x_6,x_3] + [x_5,x_2;x_4,x_1;x_6,x_3] - [x_5,x_1;x_4,x_2;x_6,x_3] \\
&+[x_5,x_3;x_2,x_1;x_6,x_4] - [x_5,x_2;x_3,x_1;x_6,x_4] + [x_5,x_1;x_3,x_2;x_6,x_4], \\
z = &\ 2[x_6,x_5;x_2,x_1;x_4,x_3] - 2[x_6,x_5;x_3,x_1;x_4,x_2] + 2[x_6,x_5;x_3,x_2;x_4,x_1] \\
&-[x_4,x_3;x_2,x_1;x_6,x_5] + [x_4,x_2;x_3,x_1;x_6,x_5] - [x_4,x_1;x_3,x_2;x_6,x_5].
\end{aligned}$$

Then

$$w(4,6) = z + 2x + y.$$

Let u_1 be the law of $F_4(\underline{\underline{N}}_6)$ obtained by interchanging x_5 and x_6 in $w(4,6)$. One checks that

$$u_1 = -z + 2y + x.$$

Hence,

$$u_2 = w(4,6) + u_1 = 3x + 3y = 3(x+y)$$

is a law of $F_4(\underline{\underline{N}}_6)$ and since this is torsion-free,

$$u_3 = x + y$$

is also a law of $F_4(\underline{\underline{N}}_6)$. From this we deduce immediately that $w(4,6) - u_3 = z + x$ is a law in $F_4(\underline{\underline{N}}_6)$ and so is $-z + y$. We put

$$u_4 = z + x,$$

and

$$w_1(4,6) = -z + y.$$

Now u_4 is obtained from $w_1(4,6)$ by interchanging the variables x_5 and x_6; also $w_1(4,6) + u_4 = u_3$ and $u_3 + u_4 = w(4,6)$, so that $w(4,6)$ is a consequence of $w_1(4,6)$.

Interchange x_4 and x_5 in $w_1(4,6)$ and denote the result by u_5. Then one checks that $u_5 - w_1(4,6)$, which is a law in $F_4(\underline{\underline{N}}_6)$ is a square in X mod $X_{(7)}$; namely

$$\begin{aligned}&-2[x_6,x_4;x_2,x_1;x_5,x_3] + 2[x_6,x_4;x_3,x_1;x_5,x_2] - 2[x_6,x_4;x_3,x_2;x_5,x_1]\\&+2[x_6,x_5;x_2,x_1;x_4,x_3] - 2[x_6,x_5;x_3,x_1;x_4,x_2] + 2[x_6,x_5;x_3,x_2;x_4,x_1]\\&+2[x_5,x_4;x_3,x_2;x_6,x_1] - 2[x_5,x_4;x_3,x_1;x_6,x_2] + 2[x_5,x_4;x_2,x_1;x_6,x_3].\end{aligned}$$

Hence the expression of which it is the square also is a law; this we denote by $w_2(4,6)$ (see (A.3.1.2),p.A.26.).

Our derivation shows the following situation: $w(4,6)$ together with the nilpotency law defines a variety in which the free group of rank 6 may contain elements of order 3 (namely every value of $x + y$ in that group) and of order 2 (namely every value of $w_2(4,6)$). We do not know whether such elements of finite order actually occur, but we do know that the group $F_4(\underline{\underline{N}}_6)$, which belongs to this variety, is torsion-free. Hence in it $x + y$ and $w_2(4,6)$ are laws, that is $w_1(4,6)$ and $w_2(4,6)$ are laws. Moreover $w(4,6)$ follows from these.

We now show that the set of laws $U = \{w_1(4,6), w_2(4,6), [x_1,x_2,\ldots,x_7]\}$ form a basis of laws for $F_4(\underline{\underline{N}}_6)$. Denote by $\underline{\underline{U}}$ the variety defined by U. Let $v \in \overline{W}^6_{6,4}$, where $\overline{W}^6_{6,4} = \{\cap \ker\varphi/\varphi : F_6(\underline{\underline{U}}) \to F_4(\underline{\underline{U}})\}$, and let v be written as a linear combination with integral coefficients of distinct basic commutators. Then (by Lemma (1.2.9)) each of these has weight 6 and (by Lemma (1.2.12)) we may further assume that v is homogeneous of weight 6 in 6 generators, that is v is a linear combination of basic commutators of weight 6 in 6 generators of 4 types: left-normed, type (3,3), type (4,2) and type (2,2,2). Assume without loss of generality that v is a linear combination of all such basic commutators. We compute relations in $F_6(\underline{\underline{U}})$ from the laws $w_1(4,6)$, $w_2(4,6)$ and eliminate the following superfluous generators (see A.3.1.3), p.A.27.-A.28.),

$$d_{578},\ d_{579},\ d_{580},\ d_{583},\ d_{604}.$$

Now write v in the form

$$v = v_1 + v_2 + v_3 + v_4,$$

where v_1 is the component of v consisting of left-normed basic commutators alone, v_2 is the component of v consisting of basic commutators of type (3,3) alone, v_3 is the component of v consisting of basic commutators of type (4,2) alone and v_4 is the component of v consisting of basic commutators of type (2,2,2) alone. Then we have:

(3.1.1) The commutators in v_1 have trivial coefficients, that is v_1 is trivial.

Proof. To begin with, we note that v_1 consists of the left-normed basic commutators d_{485}, d_{486}, d_{487}, d_{488}, d_{489}. Map v by the homomorphisms:

	φ_1	φ_2
$1 \to$	1	1
$2 \to$	1	2
$3 \to$	1	3
$4 \to$	2	4
$5 \to$	3	4
$6 \to$	4	4 .

Now under φ_1, d_{485}, d_{486}, are mapped into the trivial element while d_{487}, d_{488}, d_{489} go into distinct left-normed basic commutators in 4 generators. Since (by Lemma (3.0.1)) no other images expressed in terms of basic commutators contain left-normed basic commutators we see that (1.2.2) together with $v\varphi_1 = 0$ implies that $e_{487} = e_{488} = e_{489} = 0$.

Similarly we show that $e_{485} = e_{486} = 0$.

(3.1.2) The commutators in v_2 have trivial coefficients, that is v_2 is trivial.

Proof. It follows from Lemma (3.1.1) that $v = v_2 + v_3 + v_4$. We first show that $v_2 \in \overline{W}^6_{6,4}$. Let φ be any homomorphism of $F_6(\underline{\underline{U}})$ into $F_2(\underline{\underline{U}})$. Then $v\varphi = v_2\varphi + v_3\varphi + v_4\varphi$. Write $v\varphi$ (that is $v_2\varphi$, $v_3\varphi$ and $v_4\varphi$) in terms of distinct basic commutators. Then (1.2.2) with $v = 0$ shows that the coefficients of these commutators are trivial. Since (by Lemma (3.0.1)) $v_3\varphi$ and $v_4\varphi$ do not contain commutators of type (3.3) and $v_2\varphi$ consists of basic commutators of type (3,3) alone we must have $v_2\varphi = 0$. Now φ is arbitrary, hence $v_2 \in \overline{W}^6_{6,4}$.

Suppose now that $c_1 = [x,y,z;t,u,v]$ is any basic commutator occurring in v_2. The coefficient of this is trivial as the following argument shows. Map v_2 by the homomorphism:

$$\begin{aligned} & \quad\ \varphi \\ x &\to 4 \\ y &\to 1 \\ z &\to 1 \\ t &\to 3 \\ u &\to 2 \\ v &\to 2 \end{aligned}$$

Under φ, c_1 is mapped into $c_1' = [4,1,1;3,2,2]$. Suppose the basic commutator $c_2 = [a,b,c;d,e,f]$ occurs in v_2 and its image, when expressed in terms of basic commutators, contains c_1'. Then $[a,b,c]$ equals $[x,y,z]$ or $[t,u,v]$. For if $[a,b,c] \neq [x,y,z]$ and $[a,b,c] \neq [t,u,v]$, then in $c_2\varphi = [a\varphi,b\varphi,c\varphi;d\varphi,e\varphi,f\varphi]$, $[a\varphi,b\varphi,c\varphi] \neq [4,1,1]$ and $[a\varphi,b\varphi,c\varphi] \neq [3,2,2]$. Consequently the expansion of $c_2\varphi$ into basic commutators will not include c_1', a contradiction. Hence $[a,b,c] = [x,y,z]$ or $[a,b,c] = [t,u,v]$. Similarly $[d,e,f] = [x,y,z]$ or $[d,e,f] = [t,u,v]$. But if $[x,y,z;t,u,v]$ is basic, then $[t,u,v;x,y,z]$ is not; therefore $c_2 = c_1$, that is c_1 is the only basic commutator in v_2 whose image contains the commutator c_1'. Hence (1.2.2) with $v_2\varphi = 0$ now shows that the coefficient of c_1 is trivial.

(3.1.3) The commutators in v_3 have trivial coefficients, that is v_3 is trivial.

Proof. We have $v = v_3 + v_4$. Suppose that $c = [x,y,z,t;u,v]$ occurs in v_3 and φ is the homomorphism:

$$
\begin{array}{ccc}
 & & \varphi \\
x & \to & 2 \\
y & \to & 1 \\
z & \to & 1 \\
t & \to & 1 \\
u & \to & 4 \\
v & \to & 3.
\end{array}
$$

Then, $v\varphi = v_3\varphi + v_4\varphi$ Expand $v\varphi$ (that is $v_3\varphi$, $v_4\varphi$) into basic commutators. Under φ, c is mapped into the commutator $c_1 = [2,1,1,1;4,3]$. Now (by Lemma (3.0.1)) $v_4\varphi$ does not contain any commutator of type $(4,2)$ and hence does not contain c_1. This means that a commutator in v whose image contains c_1 must be of type $(4,2)$, that is in v_3. Furthermore, any such commutator in v_3 must have u and v as its last two entries. Hence $[x,y,z,t;u,v]$, $[z,y,x,t;u,v]$, $[t,y,x,z;u,v]$ are the only possible commutators in v_3 whose images contain c_1. Now the last two commutators have trivial images. Thus $[x,y,z,t;u,v]$ is the only commutator in v_3 (and so in $v_3 + v_4$) whose image contains c_1. Now (1.2.2) with $(v_3 + v_4)\varphi = 0$ gives immediately that the coefficient of c is trivial.

(3.1.4) The commutators in v_4 have trivial coefficients, that is v_4 and therefore v, is trivial.

Proof. We now have $v = v_4$. Since the commutators d_{578}, d_{579}, d_{580}, d_{583}, d_{604} are eliminated (see(A.3.1.3), p.A.27.-A.28.) we may assume that they are not present in v_4. Suppose $c = [6,x;y,z;5,u]$ is in v_4. We want to show first that c has trivial coefficient. Map v by the homomorphism:

$$\begin{array}{ccc} & & \varphi \\ 6 & \to & 4 \\ x & \to & 3 \\ y & \to & 2 \\ z & \to & 1 \\ 5 & \to & 2 \\ u & \to & 1. \end{array}$$

Under φ, c is mapped into the commutator $c_1 = [4,3;2,1;2,1]$. Now any commutator in v_4 whose image contains c_1 must have $6,x$ as its first and second entry respectively. The only commutator in v_4 other than c

with this property is $c' = [6,x;u,z;5,y]$. But this has trivial image, hence c is the only commutator in v_4 whose image contains c_1. Again (1.2.2) with $v_4\varphi = 0$ gives that the coefficient of c is zero. Similarly we show that the coefficient of c' is trivial. Thus we see that in v_4 commutators of the form $[6,x;y,z;t,u]$, $x \varepsilon \{1,2,3,4\}$ (note that for each $x \varepsilon \{1,2,3,4\}$ there are only two such commutators in v_4) have trivial coefficients. To show that the other commutators in v_4 have trivial coefficients, we find the following lemma useful.

(3.1.5) Lemma. Suppose $v \varepsilon \overline{W}^6_{6,4}$ and v is expanded into basic commutators. Let $x \varepsilon \{1,2,3,4\}$. If the number of commutators in v with 6 and x as their first and second entry respectively is less than or equal to 2, then their coefficients are trivial.

Proof. Without loss of generality we assume that there are two commutators say c_1, c_2 with the given property. Then if neither c_1 nor c_2 is a commutator we have eliminated, the foregoing argument gives immediately that they have trivial coefficients. If one of them say c_1 is a commutator we have eliminated, direct substitution and grouping of like-terms in the result gives again that the coefficients are zero. This proves the lemma.

We have now reduced the problem to the stage where direct application of suitable endomorphism to v_4, using as usual the fact that $\overline{W}^6_{6,4}$ is fully invariant in $F_6(\underline{\underline{U}})$, will with the aid of Lemma (3.1.5) show that all the remaining basic commutators in v_4 have trivial coefficients. The computation is given in the Appendix (see (A.3.1.4), p.A.28.-A.30.). Thus we have proved:

(3.1.6) Theorem. The set of laws $\{w_1(4,6), w_2(4,6), [x_1,x_2,\ldots,x_7]\}$ is a basis of laws for $F_4(\underline{\underline{N}}_6)$.

(3.2) The laws of $F_3(\underline{\underline{N}}_6)$.

Let $u = [w(3,5),x_6]$ where $w(3,5)$ is the law of $F_3(\underline{\underline{N}}_5)$ given in page 19 of Chapter 2. Then, by (1.2.15), u is a law of $F_3(\underline{\underline{N}}_6)$ which distinguishes $F_3(\underline{\underline{N}}_6)$ from $\underline{\underline{N}}_6$. In terms of basic commutators (see (A.3.2.1.), p.A.30),

$$
\begin{aligned}
u = &-[x_4,x_3,x_5,x_6;x_2,x_1] + [x_4,x_2,x_5,x_6;x_3,x_1] - [x_4,x_1,x_5,x_6;x_3,x_2] - \\
&[x_3,x_2,x_5,x_6;x_4,x_1] + [x_3,x_1,x_5,x_6;x_4,x_2] - [x_2,x_1,x_5,x_6;x_4,x_3] \\
&+[x_4,x_3,x_6;x_2,x_1x,_5] - [x_4,x_3,x_5;x_2,x_1,x_6] - [x_4,x_2,x_6;x_3,x_1,x_5] + \\
&[x_4,x_2,x_5;x_3,x_1,x_6] + [x_4,x_1,x_6;x_3,x_2,x_5] - [x_4,x_1,x_5;x_3,x_2,x_6].
\end{aligned}
$$

Let

$$
\begin{aligned}
w_1(3,6) = &+[x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_3,x_5;x_2,x_1,x_6] - [x_4,x_2,x_6;x_3,x_1,x_5] + \\
&[x_4,x_2,x_5;x_3,x_1,x_6] + [x_4,x_1,x_6;x_3,x_2,x_5] - [x_4,x_1,x_5;x_3,x_2,x_6], \\
w_2(3,6) = &[x_4,x_3,x_5,x_6;x_2,x_1] - [x_4,x_2,x_5,x_6;x_3,x_1] + [x_4,x_1,x_5,x_6,x_3,x_2] \\
&+[x_3,x_2,x_5,x_6;x_4,x_1] - [x_3,x_1,x_5,x_6;x_4,x_2] + [x_2,x_1,x_5,x_6;x_4,x_3].
\end{aligned}
$$

Now Lemma (3.0.1) implies that $w_1(3,6)$ and hence $w_2(3,6)$ are laws of $F_3(\underline{\underline{N}}_6)$. On the other hand, the laws $w_1(4,6) = -z+y$, $w_2(4,6)$ of $F_4(\underline{\underline{N}}_6)$ given in the appendix (A.3.1.2), p.A.26) to the present chapter are also laws of $F_3(\underline{\underline{N}}_6)$. We will prove that z, and hence y (p. 33), is actually a law of $F_3(\underline{\underline{N}}_6)$.

(3.2.1) Lemma. The word $z = 2[x_6,x_5;x_2,x_1;x_4,x_3] - 2[x_6,x_5;x_3,x_1;x_4,x_2] + 2[x_6,x_5;x_3,x_2;x_4,x_1] - [x_4,x_3;x_2,x_1;x_6,x_5] + [x_4,x_2;x_3,x_1;x_6,x_5] - [x_4,x_1;x_3,x_2;x_6,x_5]$ is a law of $F_3(\underline{\underline{N}}_6)$.

Proof. Let a_i, $i = 1,2,\ldots,6$ be arbitrary elements of $F_3(\underline{\underline{N}}_6)$. Write a_i, $i \in \{1,2,\ldots,6\}$ in the form:

$$a_i = \alpha_i(1)g_1 + \alpha_i(2)g_2 + \alpha_i(3)g_3 + c$$

where $\alpha_i(j), j=1,2,3$ are integers and c is a sum of commutators of weight greater than or equal to 2. Substitute a_i for x_i in z and denote the result by z'. Then, on expanding, linearity gives each term in z' as a sum of terms whose entries are generators. Let S_i, $i = 1,2,\ldots,6$ be the sums obtained by expanding respectively the terms $2[a_6,a_5;a_2,a_1;a_4,a_3]$, $-2[a_6,a_5;a_3,a_1;a_4,a_2]$, $2[a_6,a_5;a_3,a_2;a_4,a_1]$, $-[a_4,a_3;a_2,a_1;a_6,a_5]$, $[a_4,a_2;a_3,a_1;a_6,a_5]$, $-[a_4,a_1;a_3,a_2;a_6,a_5]$. In what follows the generator g_i will again be abbreviated to i. Let $c_1 = [i_1,i_2;i_3,i_4;i_5,i_6]$ be a commutator in S_1 and denote its coefficient by 2λ. Then $\lambda = \alpha_6(i_1)\alpha_5(i_2)\alpha_2(i_3)\alpha_1(i_4)\alpha_4(i_5)\alpha_3(i_6)$ and c_1 is the only term in S_1 whose coefficient involves precisely the same symbols $\alpha_i(j)$ as λ. Moreover, $c_2 = [i_1,i_2;i_6,i_4;i_5,i_3]$, $c_3 = [i_1,i_2;i_6,i_3;i_5,i_4]$, $c_4 = [i_5,i_6;i_3,i_4;i_1,i_2]$, $c_5 = [i_5,i_3;i_6,i_4;i_1,i_2]$, $c_6 = [i_5,i_4;i_6,i_3;i_1,i_2]$ are the only terms in $S_2,S_3,\ldots,S_6$ respectively that occur with coefficients involving these same $\alpha_i(j)$ as λ; in fact the coefficients are, in order, -2λ, 2λ, $-\lambda$, λ, $-\lambda$. Now, because only 3 generators occur altogether, just one of the following cases must occur in c_1:

(1) $i_5 = i_6$; (2) $i_5 \neq i_6$, $i_4 = i_5$; (3) $i_5 \neq i_6$, $i_4 = i_6$;

(4) $i_5 \neq i_6$, $i_4 \neq i_5$, $i_4 \neq i_6$, $i_3 = i_4$;

(5) $i_5 \neq i_6$, $i_4 \neq i_5$, $i_4 \neq i_6$, $i_3 = i_5$;

(6) $i_5 \neq i_6$, $i_4 \neq i_5$, $i_4 \neq i_6$, $i_3 = i_6$.

One checks that in every case $2c_1 - 2c_2 + 2c_3 - c_4 + c_5 - c_6 = 0$, and the common coefficient λ makes no further difference. Since the number of terms contained in S_i is the same for all $i \in \{1,2,\ldots,6\}$ and no two terms in S_i have coefficients formed from the same set of symbols $\alpha_i(j)$, if we now let c_1 run through the terms in S_1; then c_i also runs through the terms in S_i for each $i \in \{2,3, \ldots 6\}$, and $S_1 + S_2 + \ldots + S_6 = 0$ follows. Hence z' is zero and z is a law of $F_3(\underline{\underline{N}}_6)$.

Denote by $w_3(3,6)$, $w_4(3,6)$, $w_5(3,6)$ respectively the laws z, y, $w_2(4,6)$ of $F_3(\underline{\underline{N}}_6)$ and by $\underline{\underline{U}}$ the variety defined by U, where $U = \{w_1(3,6), w_2(3,6), w_3(3,6), w_4(3,6), w_5(3,6), [x_1,x_2,\ldots,x_7]\}$. We will show that U is a basis of laws for $F_3(\underline{\underline{N}}_6)$. As usual, relations in $F_6(\underline{\underline{U}})$ are obtained from the laws in U and superfluous generators eliminated; Lemma (3.0.1) is essential to these computations. Those eliminated are (see (A.3.2.2) - (A.3.2.4), p. A.31-A.38):

d_{509}, d_{510}, d_{511}, d_{520}, d_{521}, d_{522}, d_{523}, d_{528}, d_{529},
d_{545}, d_{554}, d_{557}, d_{558}, d_{566}, d_{569}, d_{570}, d_{572}, d_{573},
d_{574}, d_{575}, d_{576}, d_{577}, d_{578}, d_{579}, d_{580}, d_{581}, d_{584},
d_{587}, d_{592}, d_{590}, d_{593}, d_{596}, d_{599}.

Let $v \in \overline{W}^6_{6,3}$, where $\overline{W}^6_{6,3} = \{\cap \ker\varphi \,/\, \varphi : F_6(\underline{\underline{U}}) \to F_3(\underline{\underline{U}})\}$ and let v be written as a linear combination with integral coefficients of distinct basic commutators. Then Lemmas (1.2.13), (1.2.14) give that each commutator in v has weight 5 or 6 and involves at least 4 generators. Write v in the form:

$$v = u_1 + u_2 + u_3,$$

where u_1 is the component of v consisting of basic commutators of weight 5 in precisely 4 generators alone, u_2 is the component of v consisting of basic commutators of weight 5 in precisely 5 generators alone and u_3 is the component of v consisting of basic commutators of weight 6 alone. Then we have:

(3.2.2) The commutators in u_1 have trivial coefficients, that is u_1 is trivial.

Proof. We use the terminology that a commutator of the form $[x,y,z;t,u]$ is a commutator of type $(3,2)$. Thus u_1, u_2 consist of two kinds of commutators, namely left-normed and type $(3,2)$. First we show that the left-normed commutators in u_1 have trivial coefficients. Map v by the homomorphisms:

	φ_{11}	φ_{12}	φ_{13}
$1 \to$	1	1	1
$2 \to$	1	2	2
$3 \to$	2	2	3
$4 \to$	3	3	3
$5 \to$	0	0	0
$6 \to$	0	0	0

We need only consider the images of commutators of weights 5 and 6 in precisely the generators 1,2,3,4. Under φ_{1_i}, $i \in \{1,2,3\}$, the left-normed commutators in u_1 are mapped into basic commutators of the same type. Since the expansion of a commutator of weight 6 when expressed in terms of basic commutators does not contain commutators of weight 5, and that of a commutator of type (3,2) does not contain left-normed commutators, we can obtain equations in the coefficients of the left-normed basic commutators in u_1 by considering only the images of these commutators. They have trivial coefficients (see (A.3.2.5), p. A.38 - A.39). It follows that left-normed commutators in precisely 4 generators in u_1 have trivial coefficients. We now show that commutators of type (3,2) in u_1 also have trivial coefficients. Map v by the homomorphisms:

	φ_{21}	φ_{22}	φ_{23}	φ_{24}
$1 \to$	1	1	1	1
$2 \to$	2	2	2	[2,1]
$3 \to$	3	3	[2,1]	2
$4 \to$	[2,1]	[3,1]	3	3
$5 \to$	0	0	0	0
$6 \to$	0	0	0	0 .

Only images of basic commutators of type (3,2) in precisely the generators 1,2,3,4 need be considered. The coefficients of these commutators are trivial (see (A.3.2.6), p.A.39-A.40). Hence the commutators of type (3,2) in precisely 4 generators have trivial coefficients and we conclude that all commutators in u_1 have trivial coefficients, that is u_1 is trivial.

(3.2.3) The commutators in u_2 have trivial coefficients, that is u_2 is trivial.

Proof. We have $v = u_2 + u_3$. Map v by the homomorphism:

$$\begin{array}{lll} 1 \rightarrow & 1 & 1 \\ 2 \rightarrow & 1 & 2 \\ 3 \rightarrow & 1 & 3 \\ 4 \rightarrow & 2 & 3 \\ 5 \rightarrow & 3 & 3 \\ 6 \rightarrow & 0 & 0 \end{array} .$$

We see at once that the left-normed commutators in u_2 namely $[2,1,3,4,5]$, $[3,1,2,4,5]$, $[4,1,2,3,5]$, $[5,1,2,3,4]$ have trivial coefficients. Suppose $c_1 = [x,y,z;t,u]$ is any commutator of type (3,2) in u_2. We will show that c_1 has trivial coefficient. Map v by the homomorphism ψ which takes $t \rightarrow 2$, $u \rightarrow 1$, $x \rightarrow [3,2]$, $y \rightarrow 3$, $z \rightarrow 3$. Only images of basic commutators of type (3,2) in precisely the generators 1,2,3,4,5 in v need be considered since commutators of weight 6 are mapped into the trivial element. Under ψ,

c_1 is mapped into $c' = [3,2,3,3,;2,1]$. Now any commutator $c = [a,b,c;d,e]$ in precisely the generators $1,2,\ldots 5$ whose image under ψ when expressed in terms of basic commutators contain c' only if $[d,e] = [t,u]$. For, if $[d,e] \neq [t,u]$, then in $c\psi = [a\psi, b\psi, c\psi; d\psi, e\psi]$, $[d\psi, e\psi] \neq [2,1]$. The image $c\psi$ is either of type $(2,2,2)$ or $(4,2)$. If it is of type $(2,2,2)$, then by Lemma (3,0.1) its expansion into basic commutators consists only of commutators of the same type and so cannot contain c'. If it is of type $(4,2)$, then by Lemma (3.0.1) again, its expansion into basic commutators contains only commutators of type $(4,2)$ whose last 2 entries are $d\psi$ and $e\psi$ and so cannot contain c'. In either case, we arrive at a contradiction. Hence we must have $[d,e] = [t,u]$. Now the only other commutator of type $(3,2)$ in precisely the generators $1,2,\ldots 5$ which satisfies this condition is $[z,y,x;t,u]$. But this is mapped by ψ into the trivial element. Hence c_1 is the only such commutator whose image in terms of basic commutators contains c' when expanded. Hence (1.2.2) together with $v\psi = 0$ now gives that the coefficients of c' is trivial. This implies that in v all basic commutators of type $(3,2)$ in precisely the generators $1,2,\ldots,5$ have trivial coefficients. Hence, in v the basic commutators of type $(3,2)$ in precisely 5 generators have trivial coefficients, that is u_2 is trivial.

We may now assume that v is homogeneous of weight 6 in 6 generators. Again let v be written in the form:

$$v = v_1 + v_2 + v_3 + v_4,$$

where, v_1 is the component of v consisting of left-normed basic commutators alone, v_2 is the component of v consisting of basic commutators of type $(3,3)$ alone, v_3 is the component of v consisting of basic commutators of type $(4,2)$ alone and v_4 is the component of v consisting of basic commutators of type $(2,2,2)$ alone. We have:

(3.2.4) The commutators in v_1 have trivial coefficients, that is v_1 is trivial.

Proof. Map v by the homomorphisms:

$$\begin{array}{lll} 1 \to & 1 & 1 \\ 2 \to & 2 & 1 \\ 3 \to & 3 & 1 \\ 4 \to & 4 & 2 \\ 5 \to & 4 & 3 \\ 6 \to & 4 & 4 \end{array}$$

and obtain immediately that the commutators in v_1, namely $[2,1,3,4,5,6]$, $[3,1,2,4,5,6]$, $[4,1,2,3,5,6]$, $[5,1,2,3,4,6]$, $[6,1,2,3,4,5]$, have trivial coefficients.

(3.2.5) The commutators in v_2 have trivial coefficients, that is v_2 is trivial.

Proof. We have $v = v_2 + v_3 + v_4$. Now $v_2 \in \overline{W}^6_{6,3}$ as a consequence of Lemma (3.0.1), as in the case of the free group of rank 4. Thus we deal separately with v_2, and then with $v_3 + v_4$. First we introduce:

(3.2.6) Lemma. Suppose that $w \in \overline{W}^6_{6,3}$ and that w is a linear combination with integral coefficients of distinct basic commutators of type (3,3) in 4 generators in each of which the same one generator g_i appears three times. Then the commutators in w have trivial coefficients.

Proof. Suppose that g_i is g_4. Map w by the homomorphisms:

		φ_{31}	φ_{32}	φ_{33}
1	$\to$	1	1	1
2	$\to$	1	2	2
3	$\to$	2	2	3
4	$\to$	3	3	3
5	$\to$	0	0	0
6	$\to$	0	0	0 .

The commutators in w have trivial coefficients. (see (A.3.2.7.), p.A.41).
To show that the commutators in w have trivial coefficients in the case where $g_i \in \{g_1, g_2, g_3\}$, we map w by the endomorphism ξ of $F_6(\underline{\underline{U}})$

which takes i to 4, 4 to i and each of the other two generators into itself. Then $w\xi$ when expressed in terms of basic commutators consists of commutators in each of which the generator g_4 appears three times. Now since $\overline{W}^6_{6,3}$ is fully invariant in $F_6(\underline{\underline{U}})$, $w\xi \varepsilon \overline{W}^6_{6,3}$. Hence, by the first part of the proof, the coefficients of the commutators in the expansion of $w\xi$ are trivial; thus $w\xi$, and hence w, is again trivial. This completes the proof.

Now map v_2 by the following endomorphisms of $F_6(\underline{\underline{U}})$.

		φ_{41}	φ_{42}	φ_{43}	φ_{44}	φ_{45}	φ_{46}	φ_{47}	φ_{48}	φ_{49}	φ_{410}
1	$\to$	1	1	1	1	1	1	1	1	1	1
2	$\to$	1	1	1	1	2	2	2	2	2	2
3	$\to$	1	2	2	2	1	3	1	2	3	3
4	$\to$	2	1	3	3	1	1	3	3	3	4
5	$\to$	3	3	1	4	3	1	1	4	4	4
6	$\to$	4	4	4	1	4	4	4	2	3	4 .

Again since $\overline{W}^6_{6,3}$ is fully invariant in $F_6(\underline{\underline{U}})$, $w\varphi_{4i} \varepsilon \overline{W}^6_{6,3}$ for each $i \varepsilon \{1,2,\ldots,10\}$. Write $w\varphi_{4i}$, $i \varepsilon \{1,2,\ldots,10\}$ as a linear combination with integral coefficients of distinct basic commutators. Lemma (3.2.6) now gives that these coefficients are trivial. The resulting relations between the coefficients of the commutators in v_2 show that these also are trivial (see (A.3.2.8.), p.A.42–A.45).

(3.2.6) The commutators in v_3 have trivial coefficients, that is v_3 is trivial.

Proof. Now we have $v = v_3 + v_4$. Since the commutators

$d_{574} = [4,1,2,3;6,5]$, $d_{573} = [3,1,2,4;6,5]$, $d_{570} = [3,1,2,5;6,4]$,
$d_{558} = [3,1,2,6;5,4]$, $d_{572} = [2,1,3,4;6,5]$, $d_{569} = [2,1,3,5;6,4]$,
$d_{557} = [2,1,3,6;5,4]$, $d_{566} = [2,1,4,5;6,3]$, $d_{554} = [2,1,4,6;5,3]$,
$d_{545} = [2,1,5,6;4,3]$

are eliminated (see (A.3.2.3), p.A. 33-A. 35), we may assume that they are not included in v_3. Before this elimination, there are, corresponding to every set of 3 generators $\{y,z,t\}$, $y < z < t$, $y \leq 3$, at most 3 basic commutators in v_3 having y,z,t as their second, third and fourth entry respectively. Suppose that $\{x,u,v\} = \{1,2,\ldots,6\} \setminus \{y,z,t\}$ and $x > u > v$. Then for $y = 1$, the commutators are $[x,y,z,t;u,v]$, $[u,y,z,t;x,v]$ and $[v,y,z,t;x,u]$; for $y = 2$, the commutators are $[x,y,z,t;u,v]$ and $[u,y,z,t;x,v]$; for $y = 3$, the commutator is $[x,y,z,t;u,v]$. As a result of the elimination of the commutator $[v,y,z,t;x,u]$ for $y = 1$, we have that for this value of y, the commutators are $[x,y,z,t;u,v]$ and $[u,y,z,t;x,v]$.
Now let $c_1 = [x,1,z,t;u,v]$ be any commutator in v_3 whose second entry is 1 and assume without loss of generality that $x > u > v$. We will show that the coefficient of this commutator is trivial. Map v by the homomorphism φ which takes $1,z,t$ into 1 and x,u,v into 2,3,2 respectively. Under φ, c_1 is mapped into $c_1' = [2,1,1,1;3,2]$.

Now by Lemma (3.0.1) any commutator in v whose image when expanded into basic commutators contains c_1' is in v_3. But in v_3 a commutator has this property only if it contains $1,z,t$ as three of its first 4 entries. The only possible such commutators in v_3 are $[u,1,z,t;x,v]$ and those which are of the form $[z,1,.,.;.,.]$, $[t,1,.,.;.,.]$. But these are mapped by φ into the trivial element. Hence c_1 is the only commutator in v_3 (and hence in v) whose image when expanded into basic commutators contain c_1'. Thus (1.2.2) with $v\varphi = 0$ now shows that the coefficient of c_1 is trivial. Therefore we conclude that commutators of the form $[x,1,z,t;u,v]$ have trivial coefficients. We may now assume that they are not included in v_3. Let $c_2 = [x,2,z,t;u,v]$ be any commutator in v_3 whose second entry is 2. Again assume that $x > u > v$. This implies that $v = 1$. Map v by the homomorphism ψ which takes $2,z,t$ to 1 and x,u,v to $2,3,2$ respectively. Under ψ, c_2 is mapped into $c_2' = [2,1,1,1;3,2]$, $[u,2,z,t;x,v]$ is another commutator in v_3 whose image when expanded into basic commutators contain c_2'. Any other commutators in v_3 having this property must be of the form: $[z,2,.,.;.,.]$, $[t,2,.,.;.,.]$, $[2,v,z,t;x,u]$. The first two are mapped into the trivial element. The last one is not present in v_3 by the first part of the proof. Hence c_2 is the only commutator in v_3 whose image when expanded into basic commutators contain c_2'. Thus the commutator $[x,2,z,t;u,v]$ has trivial coefficient, and so we conclude that in v_3 commutators of the form $[x,2,z,t;u,v]$ have trivial coefficients. We may now assume that they are not present in v_3. Finally, let $c_3 = [x,3,z,t;u,v]$ be any commutator in v_3 whose second entry is 3. Assume again that $x > u > v$. Map v by the homomorphism Θ which takes $3,z,t$ to 1 and x,u,v to $2,3,2$ respectively. Under Θ, c_3 is mapped into $c_3' = [2,1,1,1;3,2]$. As before we can show that c_3 is the only commutator in v_3 whose image when expanded into basic commutators contains c_3'. Thus c_3 has trivial coefficient and we conclude that in v_3, commutators of the form $[x,3,z,t;u,v]$ have trivial coefficients. Thus, all commutators in v_3 have trivial coefficients, that is v_3 is trivial.

(3.2.7) <u>The commutators in</u> v_4 <u>have trivial coefficients, that is</u> v_4 <u>is trivial</u>.

Proof. We have $v = v_4$. Since some commutators of type $(2,2,2)$ have been eliminated (see (A.3.2.4), p.A.35–A.38), we may assume that they are not included in v_4, that is v_4 is a linear combination with integral coefficients of the 16 commutators listed in the appendix (see (A.3.2.9), p.A.46). Map v by the homomorphisms

	φ_{51}	φ_{52}	φ_{53}
$1 \to$	1	1	1
$2 \to$	1	2	2
$3 \to$	2	1	2
$4 \to$	2	2	1
$5 \to$	2	2	2
$6 \to$	3	3	3 .

Under φ_{51}, $c_1 = [6,3;4,1;5,2]$ is mapped into $c_1' = [3,2;2,1;2,1]$ and is the only commutator in v_4 whose image when expanded into basic commutators contains c_1'. Hence (1.2.2) with $v\varphi_{41} = 0$ gives that the coefficient of c_1, namely e_{582}, is trivial. Similarly using the homomorphisms φ_{52}, φ_{53}, we can show that the coefficients e_{585}, e_{588} are trivial. Now map v by the homomorphisms:

	φ_{61}	φ_{62}	φ_{63}	φ_{64}	φ_{65}	φ_{66}	φ_{67}	φ_{68}	φ_{69}	φ_{610}	φ_{611}
$1 \to$	1	1	1	1	1	2	1	2	1	1	2
$2 \to$	1	1	2	1	2	1	1	1	2	2	1
$3 \to$	1	1	2	2	2	1	2	2	1	3	2
$4 \to$	2	2	2	3	3	2	2	1	2	1	1
$5 \to$	3	2	3	3	3	3	3	3	3	2	2
$6 \to$	3	3	3	3	3	2	2	2	2	2	3 .

Again we obtain equations in the coefficients of the commutators in v_4. We solve these and obtain that e_{591}, e_{594}, e_{595}, e_{597}, e_{600} are trivial, and that $e_{583} = e_{589}$, $e_{586} = -e_{589}$; $e_{595} = e_{601}$, $e_{598} = -e_{601}$; $e_{589} = -e_{601}$ (see (A.3.2.10), p.A.46-A.49) so that v_4 can now be written in the form:

$$\begin{aligned} v_4 = e_{601}\{&-[6,1;4,3;5,2] + [6,1;4,2;5,3] - [6,1;3,2;5,4] \\ &+[5,1;4,3;6,2] - [5,1;4,2;6,3] + [5,1;3,2;6,4]\} \\ &+e_{602}[4,3;2,1;6,5] + e_{603}[4,2;3,1;6,5]. \end{aligned}$$

Now map v_4 by the homomorphisms:

	φ_{71}	φ_{72}	φ_{73}
$1 \rightarrow$	1	1	1
$2 \rightarrow$	2	2	1
$3 \rightarrow$	2	3	3
$4 \rightarrow$	3	2	2
$5 \rightarrow$	1	1	1
$6 \rightarrow$	2	2	3 .

We check that

(1) Under φ_{71}, $v_4\varphi_{71} = e_{602}[3,2;2,1;2,1] + e_{603}[3,2;2,1;2,1]$, so that $e_{602} + e_{603} = 0$;

(2) Under φ_{72}, $v_4\varphi_{72} = -e_{602}[3,2;2,1;2,1]$, so that $e_{602} = 0$ and so $e_{603} = 0$ by (1).

(3) Under φ_{73}, $v_4\varphi_{73} = -e_{601}[3,1;2,1;3,1]$ so that $e_{601} = 0$.

Thus, the commutators in v_4 have trivial coefficients and the proof is complete. We have:

(3,2.8) Theorem. The set of laws $\{w_1(3,6), w_2(3,6), \ldots, w_5(3,6), [x_1, x_2, \ldots, x_7]\}$ is a basis of laws for $F_3(\underset{=}{N}_6)$.

CHAPTER 4

THE LAWS OF THE FREE GROUP $F_2(\underline{\underline{N}}_6)$

The basis of laws that we have found in this case consists of a homogeneous law of weight 5 in 5 variables and several homogeneous laws of weight 6 in 6 variables. The homogeneous law of weight 5 in 5 variables is actually the law $w(3,5)$ (Chapter 2, p.19). We leave the interpretation of this fact to the concluding chapter and concentrate now entirely on establishing a basis of laws for the free group of class 6 and rank 2.

We obtain immediately (by (1.2.15), p.18) from the laws $w_1(2,5)$, $w_2(2,5)$ of $F_2(\underline{\underline{N}}_5)$ the following laws that distinguish $F_2(\underline{\underline{N}}_6)$ from $\underline{\underline{N}}_6$.

$$\mu = [[x_1,x_2;x_3,x_4;x_5],x_6],$$
$$\nu = [[x_2,x_1,x_5;x_4,x_3] + [x_3,x_2,x_5;x_4,x_1] - [x_3,x_1,x_5;x_4,x_2],x_6].$$

In terms of basic commutators (see (A.4.1.1), (A.4.1.2), p.A.50), they reduce to:

$$\begin{aligned}\mu = &-[x_4,x_3,x_5;x_2,x_1,x_6] - [x_4,x_3,x_6;x_2,x_1,x_5] + [x_2,x_1,x_5,x_6;x_4,x_3]\\ &-[x_4,x_3,x_5,x_6;x_2,x_1],\\ \nu = &\ [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_2,x_6;x_3,x_1,x_5] + [x_4,x_1,x_6;x_3,x_2,x_5]\\ &-[x_3,x_2,x_5,x_6;x_4,x_1] + [x_3,x_1,x_5,x_6;x_4,x_2] - [x_2,x_1,x_5,x_6;x_4,x_3].\end{aligned}$$

From these we deduce by using Lemma (3.0.1) that

$$w_1(2,6) = [x_4,x_3,x_5;x_2,x_1,x_6] + [x_4,x_3,x_6;x_2,x_1,x_5],$$

$$w_2(2,6) = [x_2,x_1,x_5,x_6;x_4,x_3] - [x_4,x_3,x_5,x_6;x_2,x_1],$$

$$w_3(2,6) = [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_2,x_6;x_3,x_1,x_5] + [x_4,x_1,x_6;x_3,x_2,x_5],$$

$$w_4(2,6) = -[x_3,x_2,x_5,x_6;x_4,x_1] + [x_3,x_1,x_5,x_6;x_4,x_2] - [x_2,x_1,x_5,x_6;x_4,x_3],$$

are laws that distinguish $F_2(\underline{\underline{N}}_6)$ from $\underline{\underline{N}}_6$.

Now every commutator $[a,b;c,d;e,f]$ in the generators of $F_2(\underline{\underline{N}}_6)$ is trivial since $[a,b] = \pm[c,d]$. Hence

$$w_5(2,6) = [x_1,x_2;x_3,x_4;x_5,x_6]$$

is another homogeneous law of weight 6 in 6 variables that is not a law in $\underline{\underline{N}}_6$.

In trying to show that the set of laws $\{w_1(2,6), w_2(2,6),\ldots,w_5(2,6), [x_1,x_2,\ldots,x_7]\}$ form a basis of laws for $F_2(\underline{\underline{N}}_6)$ we met with difficulties the nature of which indicated that the following may also be laws:

$$w_6(2,6) = -[x_6,x_4,x_5;x_2,x_1,x_3] + [x_5,x_3,x_6;x_2,x_1,x_4],$$

$$\begin{aligned} w_7(2,6)=w(3,5) = &-[x_4,x_1,x_5;x_3,x_2] + [x_4,x_2,x_5;x_3,x_1] - [x_4,x_3,x_5;x_2,x_1] \\ &+[x_3,x_1,x_5;x_4,x_2] - [x_3,x_2,x_5;x_4,x_1] - [x_2,x_1,x_5;x_4,x_3]. \end{aligned}$$

We will presently show that they are indeed laws of $F_2(\underline{\underline{N}}_6)$.

(4.1) Lemma. The word $w_6(2,6) = -[x_6,x_4,x_5;x_2,x_1,x_3] + [x_5,x_3,x_6;x_2,x_1,x_4]$ is a law of $F_2(\underline{\underline{N}}_6)$.

Proof. Let a_i, $i = 1,2,\ldots,6$, be arbitrary elements of $F_2(\underline{\underline{N}}_6)$. Write a_i, $i \in \{1,2,\ldots,6\}$, in the form:

$$a_i = \alpha_i(1)g_1 + \alpha_i(2)g_2 + c_i,$$

where $\alpha_i(j)$, $j = 1,2,3$ are integers and c_i is a sum of commutators of weight greater than or equal to 2. Substitute a_i for x_i in $w_5(2,6)$ and denote the result by $w_5'(2,6)$. Then, on expanding, linearity gives each term in $w_5'(2,6)$ as a sum of terms whose entries are generators. Let S_1, S_2 be the sums obtained by expanding respectively the terms $-[a_6,a_4,a_5;a_2,a_1,a_3]$, $[a_5,a_3,a_6;a_2,a_1,a_4]$ of $w_5'(2,6)$. If we omit the trivial commutators in S_1 and S_2, then for $k \in \{1,2\}$, S_k can be written in the form:

$$\begin{aligned} S_k = {} & \lambda_{k1}[2,1,2;2,1,1] + \lambda_{k2}[1,2,2;2,1,1] + \lambda_{k3}[2,1,2;1,2,1] \\ & +\lambda_{k4}[1,2,2;1,2,1] + \lambda_{k5}[2,1,1;2,1,2] + \lambda_{k6}[1,2,1;2,1,2] \\ & +\lambda_{k7}[2,1,1;1,2,2] + \lambda_{k8}[1,2,1;1,2,2], \end{aligned}$$

where $\lambda_{k\ell}$, $\ell = 1,2,\ldots,8$ are products of the symbols $\alpha_i(j)$. One finds that $\lambda_{11} = -\lambda_{21}$; $\lambda_{12} = -\lambda_{25}$; $\lambda_{13} = -\lambda_{23}$; $\lambda_{14} = -\lambda_{24}$; $\lambda_{15} = -\lambda_{22}$; $\lambda_{16} = -\lambda_{26}$; $\lambda_{17} = -\lambda_{27}$; $\lambda_{18} = -\lambda_{28}$. We check that if in $S_1 + S_2$, we collect the terms together whose coefficients involve the same set of symbols $\alpha_i(j)$, each collection of commutators adds up to the trivial element so that $S_1 + S_2$ is trivial. Hence $w_5'(2,6)$ is trivial and $w_5(2,6)$ is a law of $F_2(\underline{\underline{N}}_6)$. This completes the proof.

(4.2) Lemma. The word $w_7(2,6) = -[x_4,x_1,x_5;x_3,x_2] + [x_4,x_2,x_5;x_3,x_1] - [x_4,x_3,x_5;x_2,x_1] + [x_3,x_1,x_5;x_4,x_2] - [x_3,x_2,x_5;x_4,x_1] - [x_2,x_1,x_5;x_4,x_3]$ is a law of $F_2(\underline{\underline{N}}_6)$.

Proof. Let a_i, $i = 1,2,\ldots,5$ be arbitrary elements of $F_2(\underline{\underline{N}}_6)$. Write a_i, $i \in \{1,2,\ldots,5\}$, in the form:

$$a_i = \alpha_i(1)g_1 + \alpha_i(2)g_2 + \gamma_i[g_2,g_1] + c_i,$$

where $\alpha_i(1)$, $\alpha_i(2)$, γ_i are integers and c_i is a sum of commutators of weight greater than or equal to 3. As $w_7(2,6) = w(3,5)$ is a law in the 2-generator group of class five, we know that the expansion of the expression obtained by substituting a_i for x_i must fall into the lowest term, that is it must consist of commutators of weight 6 in the generators. Therefore we need only show that these disappear.

Put $b_i = \alpha_i(g_1) + \alpha_i(g_2)$. Let $w_7'(2,6)$ be the result of substituting a_i for x_i in $w_7(2,6)$; s be the result of substituting b_i for x_i in $w_7(2,6)$; s_{1j}, $j \in \{1,2,\ldots,5\}$ be the result of substituting in $w_7(2,6)$ $\gamma_j[g_2,g_1]$ for x_j and b_i for x_i whenever $i \neq j$. Then, on expanding, linearity gives

$$w_7'(2,6) = s + s_{11} + s_{12} + \ldots + s_{15}.$$

We will first show that s_{1i} is trivial for all $i \in \{1,2,\ldots,5\}$. The generator g_i is again abbreviated to i. Now

$$s_{13} = [b_2,b_1,b_5;b_4,[2,1]] - [b_4,b_1,b_5;b_2,[2,1]] + [b_4,b_2,b_5;b_1,[2,1]]$$
$$+[[2,1],b_2,b_5;b_4,b_1] - [[2,1],b_1,b_5;b_4,b_2] - [[2,1],b_4,b_5;b_2,b_1].$$

On expanding, linearity gives each term in s_{13} as a sum of terms in each of which one entry is $[2,1]$ and the rest of the 5 entries are single generators. Let S_i, $i = 1,2,\ldots,6$ be the sums obtained by expanding respectively the terms $[b_2,b_1,b_5;b_4,[2,1]]$, $-[b_4,b_1,b_5;b_2,[2,1]]$, $+[b_4,b_2,b_5;b_1,[2,1]]$, $[[2,1],b_2,b_5;b_4,b_1]$, $-[[2,1],b_1,b_5;b_4,b_2]$, $-[[2,1],b_4,b_5;b_2,b_1]$. Let $c_1 = [i_1,i_2,i_3;i_4,[2,1]]$ be a commutator in S_1 and denote its coefficient by λ. Then $\lambda = \gamma_3\alpha_2(i_1)\alpha_1(i_2)\alpha_5(i_3)\alpha_4(i_4)$ and c_1 is the only term in S_1 whose coefficient involves precisely the same symbols $\alpha_i(j)$ as λ. Moreover, $c_2 = [i_4,i_2,i_3;i_1,[2,1]]$, $c_3 = [i_4,i_1,i_3;i_2,[2,1]]$, $c_4 = [[2,1],i_1,i_3;i_4,i_2]$, $c_5 = [[2,1],i_2,i_3;i_4,i_1]$, $c_6 = [[2,1],i_4,i_3;i_1,i_2]$ are the only terms in $S_2,S_3,\ldots,S_6$ respectively that occur with coefficients involving these same $\alpha_i(j)$ as λ; in fact the coefficients are, in order, $-\lambda$, λ, λ, $-\lambda$, $-\lambda$. Now because only 2 generators occur altogether, just one of the following cases must occur in c_1: (1) $i_1 = i_2$; (2) $i_1 \neq i_2$, $i_3 = i_1$, $i_4 = i_1$; (3) $i_1 \neq i_2$, $i_3 = i_1$, $i_4 = i_2$; (4) $i_1 \neq i_2$, $i_3 = i_2$, $i_4 = i_1$; (5) $i_1 \neq i_2$, $i_3 = i_2$, $i_4 = i_2$. We check that in every case, $c_1 - c_2 + c_3 + c_4 - c_5 - c_6 = 0$; the common coefficient λ makes no further difference. Hence, as in Lemma (3.2.1), p.42, we conclude that $S_1 + S_2 + \ldots + S_6 = 0$. Now when the pair of variables x_3, x_1 then x_3, x_2 then x_3, x_4 are interchanged in $w_7(2,6)$, the result in each case is $-w_7(2,6)$. This means that s_{1j}, $j \in \{1,2,4\}$, may be obtained from s_{13} by mapping $\gamma_3 \to \gamma_j$, $\alpha_1 \to \alpha_j$, $\beta_1 \to \beta_j$ and changing the sign of the resulting expression. Since s_{13} is trivial, we conclude that s_{1i}, $i \varepsilon \{1,2,4\}$ is trivial. It remains therefore now to show that s_{15} is trivial. Again we expand each term in s_{15} into commutators of the kind in which one entry is $[2,1]$ and the rest of the five entries are single generators. A similar argument to that used in showing that s_{13} is trivial shows that s_{15} is trivial.

Thus $w_7'(2,6) = s$. Let $t_1, t_2, \ldots, t_6$ respectively be the terms (of s) $[b_2,b_1,b_5;b_4,b_3]$, $-[b_4,b_1,b_5;b_2,b_3]$, $[b_4,b_2,b_5;b_1,b_3]$, $[b_3,b_2,b_5;b_4,b_1]$, $-[b_3,b_1,b_5;b_4,b_2]$, $-[b_3,b_4,b_5;b_2,b_1]$. Expand the entry b_j in the terms t_i of s, and keep the other entries fixed. Each expansion consists of 3 commutators, one of them of weight 6. All of them have the property that 4 of their 5 or 6 entries are precisely those that have been fixed. Let s_{2j} be the sum of all these commutators of weight 6 occurring in the expansion of the entry b_j in the terms t_i of s. Let T_1 be the sum of all terms of the form $[x,y,z;t,u]$ where $x \varepsilon \{\alpha_2(1)1, \alpha_2(2)2\}$, $y \varepsilon \{\alpha_1(1)1, \alpha_1(2)2\}$, $z \varepsilon \{\alpha_5(1)1, \alpha_5(2)2\}$, $t \varepsilon \{\alpha_4(1)1, \alpha_4(2)2\}$, $u \varepsilon \{\alpha_3(1)1, \alpha_3(2)2\}$; that is, T_1 gathers together all those commutators of weight 5 that arise, in pairs, from the expansion of t_1 with respect to each entry b_j, $j \varepsilon \{1,2,3,4,5\}$. $T_2, T_3, \ldots, T_6$ are similarly defined. Then we have:

$$s = T_1 + T_2 + \ldots + T_6 + s_{21} + s_{22} + \ldots + s_{26}.$$

As in the first part of this proof, we can show that s_{23}, s_{25} are trivial and then deduce that s_{2j}, $j \in \{1,2,4\}$ are trivial. The argument is almost the same as before and will not be repeated. Hence,

$$s = T_1 + T_2 + \ldots + T_6.$$

The following is a list of all the non-trivial commutators (not necessarily basic) of type $(3,2)$ in precisely the generators 1, 2.

$$c_1 = [2,1,1;2,1],\quad c_2 = [1,2,1;2,1],\quad c_3 = [1,2,2;2,1],$$
$$c_4 = [2,1,2;2,1],\quad c_5 = [2,1,1;1,2],\quad c_6 = [1,2,1;1,2],$$
$$c_7 = [1,2,2;1,2],\quad c_8 = [2,1,2;1,2].$$

We recall that the entries in the commutators of T_j are multiples of the generators 1 and 2. Denote by $c_i(T_j)$ that commutator in T_j whose entries taken in order are, but for the coefficients, the same as those in c_i. Then,

$$T_j = c_1(T_j) + c_2(T_j) + ,\ldots, + c_8(T_j).$$

These terms $c_i(T_j)$ now have to be expanded in multiples of the commutators c_i, and this introduces again commutators of weight 6.

In order to prove that each commutator of weight 6 appears with total coefficient zero in the final expansion, we first expand each term $c_i(T_j)$ with respect to the coefficient $\alpha_3(1)$ of the generator 1, or $\alpha_3(2)$ of the generator 2 (one, and only one, of these must appear in each $c_i(T_j)$). Each expansion produces just one commutator of weight 6 with a certain coefficient. These are shown in a table in the Appendix (see(A.4.1.3),p.A.51-55) from which it can be read off that the commutators of weight six due to this first step cancel out amongst themselves.

We have not recorded in the appendix the resulting terms of weight five which are, of course, needed for the next expansion which is conveniently taken to be that with respect to $\alpha_4(1)$ and $\alpha_4(2)$. All the calculations have, in fact, been carried out and show that in each step the resulting terms of weight six cancel. I have not been able to find a less computational and complicated proof for $w_7(2,6)$ to be a law in $F_6(\underline{\underline{N}}_6)$.

We now proceed to show that the set of laws $U = \{w_1(2,6), w_2(2,6), \ldots, w_7(2,6), [x_1, x_2, \ldots, x_7]\}$ form a basis of laws for $F_2(\underline{\underline{N}}_6)$. Let $\underline{\underline{U}}$ be the variety defined by U. Let $v \in \overline{W}^6_{6,2}$, where $\overline{W}^6_{6,2} = \{\cap \ker\varphi/\varphi : F_6(\underline{\underline{U}}) \to F_2(\underline{\underline{U}})\}$. Write v as a linear combination with integral coefficients of distinct basic commutators each of weight 5 or 6 and each involves at least 3 generators. In Chapter 2 we obtained relations from the law $w_7(2,6)$ and eliminated $[2,1,5;4,3]$, $[2,1,4;5,3]$, $[2,1,3;5,4]$, $[3,1,2;5,4]$ (see (A.2.1.2), p.A.12–A.14). We now use these again to eliminate these same commutators modulo terms of weight 6 and also use similar relations to eliminate, modulo terms of weight 6, the two commutators $[3,1,4;2,1]$ and $[3,2,4;2,1]$ in only 4 generators (see (A.4.1.4), p.A.54). Likewise we obtain relations from the laws $w_1(2,6)$, $w_2(2,6)$, $w_3(2,6)$, $w_4(2,6)$, $w_6(2,6)$ and eliminate from them some commutators of types (3,3) and (4,2) in precisely six generators, (see (A.4.1.8)-(A.4.1.12), p.A.64-A.72). Therefore we may assume that the six commutators of weight 5 and those commutators of types (3,3) and (4,2) which have been eliminated are not present in v. We will show that v is trivial. We first show that in v, commutators of weight 5 in precisely 3 generators have trivial coefficients. Map v by the homomorphisms:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9
$1 \to$	1	1	[2,1]	1	-1	1	1	1	-1
$2 \to$	2	[2,1]	1	1	1	2	2	2	2
$3 \to$	[2,1]	2	2	2	2	2	-2	1	1
$4 \to$	0	0	0	0	0	0	0	0	0
$5 \to$	0	0	0	0	0	0	0	0	0
$6 \to$	0	0	0	0	0	0	0	0	0

.

Since under φ_i, $i \in \{1,2,\ldots,9\}$, basic commutators involving the generators 4,5 or 6 are mapped into the trivial element we need only consider the images of commutators in v of weights 5 and 6 in precisely the generators 1,2,3. Here we want to obtain equations in the coefficients of basic commutators of weight 5 alone and so we consider only the images of basic commutators of weight 5. Commutators of weight 6 are omitted in the table of images (see (A.4.1.5), p.A. 55-A. 58) as they are now irrelevant. In v, the commutators of weight 5 in precisely the generators 1,2,3 have trivial coefficients, (see (A.4.1.5), p.A.55 -A.61). Hence in v, the commutators of weight 5 in precisely 3 generators have trivial coefficients. We put this fact in the following form, because it will be referred to again later on.

(4.2) <u>Lemma</u>. Suppose $v \in \overline{W}^6_{6,2}$ and v is a linear combination with integral coefficients of distinct basic commutators of weights 5 and 6 in precisely 3 generators. Then those commutators of weight 5 have trivial coefficients.

We now show that in v the commutators of weight 5 in precisely the generators $1,2,\ldots,5$ have trivial coefficients. Map v by the endomorphism ψ which maps 6 to 0 and i to i for all $i \in \{1,2,\ldots,5\}$. Then, since $\overline{W}^6_{6,2}$ is fully invariant in $F_6(\underline{\underline{U}})$, $v\psi \in \overline{W}^6_{6,2}$. We see that $v\psi$ consists of basic commutators of weights 5 and 6 in the generators $1,2,\ldots,5$. Write $v\psi$ in the form:

$$v\psi = \{v_5^{(1234)}+v_6^{(1234)}\} + \{v_5^{(1235)}+v_6^{(1235)}\} + \{v_5^{(1245)}+v_6^{(1245)}\} +$$

$$\{v_5^{(1345)}+v_6^{(1345)}\} + \{v_5^{(2345)}+v_6^{(2345)}\} + \{v_5^{(12345)}+v_6^{(12345)}\},$$

where

$$v_5^{(ijk\ell)}, v_6^{(ijk\ell)}, (ijk\ell) \in \{(1234),(1235)(1245),(1345),(2345)\},$$

are the components of v consisting respectively of commutators of weights 5 and 6 in precisely the generators i,j,k,ℓ alone, and $v_5^{(12345)}$ and $v_6^{(12345)}$ are the components of v consisting respectively of commutators of weights 5 and 6 in precisely the generators 1,2,3,4,5. The endomorphism which maps 1 to i, 2 to j, 3 to k, 4 to ℓ, 5 and 6 to 0 shows at once that $v_5^{(ijk\ell)}+v_6^{(ijk\ell)} \in \overline{W}_{6,2}^6$ for every one of the 5 choices of $(ijk\ell)$. Hence $v_5^{(12345)}+v_6^{(12345)} \in \overline{W}_{6,2}^6$. Put $z_1 = v_5^{(1234)}+v_6^{(1234)}$, $z_2 = v_5^{(12345)}+v_6^{(12345)}$. Map z_2 by the mappings φ_i, $i = 1,2,\ldots,8$ (Chapter 2, p.21), regarded now as endomorphisms of $F_6(\underline{\underline{U}})$. Now $z_1\varphi_i \in \overline{W}_{6,2}^6$ for all $i \in \{1,2,\ldots,8\}$. We write $z_2\varphi_i$, $i \in \{1,2,\ldots,8\}$, in terms of basic commutators. Then, by Lemma (4.2), those of weight 5 have trivial coefficients. Thus we obtain a set of equations in the coefficients of the commutators of weight 5 in z_2. In fact, this is precisely the same set of equations that we obtained previously (see (A.2.1.3), p.A.14–A.19). Hence as before, we conclude that the commutators of weight 5 in z_2 have trivial coefficients. Therefore in v, commutators of weight 5 in precisely 5 generators have trivial coefficients. Again we record in the form of a lemma the part that we need again later.

(4.3) Lemma. Suppose that $v \in \overline{W}_{6,2}^6$ and that v is a linear combination with integral coefficients of distinct basic commutators of weights 5 and 6 in precisely the generators $1,2,\ldots,5$. Then, provided that the 4 commutators of weight 5 that were eliminated do not occur in v, then all commutators of weight 5 in v have trivial coefficients.

To show that in v commutators of weight 5 have trivial coefficients, it thus remains to show that commutators of weight 5 in precisely 4 generators have trivial coefficients. Write z_1 in the form:

$$z_1 = v_5^{(1'234)}+v_5^{(12'34)}+v_5^{(123'4)}+v_5^{(1234')}+v_6^{(1234)},$$

where $v_5^{(1'234)}$, $v_5^{(12'34)}$, $v_5^{(123'4)}$, $v_5^{(1234')}$ are respectively the components of $v_5^{(1234)}$ in each commutator of which the generators 1,2,3,4 appear twice. Let $v_5^{(1'235)}$, $v_5^{(12'35)}$, $v_5^{(123'5)}$, $v_5^{(1235')}$, $v_6^{(1235)}$ be the result of substituting 5 for 4 in $v_5^{(1'234)}$, $v_5^{(12'34)}$, $v_5^{(123'4)}$, $v_5^{(1234')}$, $v_6^{(1234)}$ respectively and let x comprises all the terms that arise from substituting for any <u>one</u> of the repeated generators 4 in $v_5^{(1234')}$. Then if we substitute $4 + 5$ for 4 throughout in z_1 and expand so that the entries of commutators are again single generators, we have:

$$z_1 = v_5^{(1'234)}+v_5^{(12'34)}+v_5^{(123'4)}+v_5^{(1234')}+v_6^{(1234)}+v_5^{(1'235)}+v_5^{(123'5)}+v_5^{(1235')}$$
$$+v_6^{(1235)} + x + y,$$

where y is the sum of commutators of weight 6 arising from the expansion of commutators of weight 5 in z_1 after the substitution of $4 + 5$ for 4. Since $z_1 \varepsilon \overline{W}^6_{6,2}$, the result of substituting 5 for 4 in z_1 is in $\overline{W}^6_{6,2}$. Therefore $x + y \varepsilon \overline{W}^6_{6,2}$. Now x consists of basic commutators of weight 5 in precisely the generators $1,2,\ldots,5$ and y consists of commutators of weight 6 in precisely the generators $1,2,\ldots,5$. Rewrite x using the relations (r.1) - (r.4) modulo weight 6 (see (A.2.1.2.), p.A.12-A.14) so that it does not contain any of the commutators of weight 5 that we have eliminated. Lemma (4.3) now implies that the commutators in x have trivial coefficients. We use this fact to deduce that $v_5^{(1234')}$ is trivial (see (A.4.1.6), p.A.61–A.62). Hence z_1 reduces to

$$z_1 = v_5^{(1'234)}+v_5^{(12'34)}+v_5^{(123'4)}+v_6^{(1234)}.$$

We can now deduce that the second and the third terms in z_1 are trivial. To show that $v_5^{(123'4)}$ is trivial we map z_1 by the automorphism $\xi : 1 \to 1,\ 2 \to 2,\ 3 \to 4,\ 4 \to 3,\ 5 \to 5,\ 6 \to 6.$ Then,

$$z_1\xi = v_5^{(1'234)}\xi+v_5^{(12'34)}\xi+v_5^{(123'4)}\xi+v_6^{(1234)}.$$

Write each of the summands in terms of basic commutators. Now the expansion of $v_5^{(123'4)}\xi$ into basic commutators does not contain commutators of weight 6. To see, look at the commutators involving two entries '4' in (A.4.1.6.), p.A.61. After interchanging 3 and 4, each one is either basic or converted to a basic one by interchanging the entries in the first two, or in the last two places. Hence by the foregoing argument $v_5^{(123'4)}\xi$ is trivial. Now ξ is an automorphism and so we must have that $v_5^{(123'4)}$ is trivial. So now we have $z_1 = v_5^{(1'234)}+v_5^{(12'34)}+v_6^{(1234)}$. Similarly, to show that $v_5^{(12'34)}$ is trivial, we map z_1 by the automorphism $\zeta : 1 \to 1,\ 2 \to 4,\ 3 \to 3,\ 4 \to 2,\ 5 \to 5,\ 6 \to 6.$ Then,

$$z_1\zeta = v_5^{(1'234)}\zeta+v_5^{(12'34)}\zeta+v_6^{(1234)}\zeta.$$

Again we write $z_1\zeta$ in terms of basic commutators. Because the commutator $[3,2,4;2,1]$ is not included in v and so is not included in $v_5^{(12'34)}$. But this is the only commutator involving two entries '2', which after interchanging 2 and 4 leads to commutators of weight 6 when expanded into basics. Therefore the expansion of $v_5^{(12'34)}$ into basic commutators does not contain any commutators of weight 6. Hence by the foregoing argument again, $v_5^{(12'34)}\zeta$ is trivial and so $v_5^{(12'34)}$ is trivial. We therefore have $z_1 = v_5^{(1'234)}+v_6^{(1234)}$. Map z_1 by the endomorphisms:

		φ_{21}	φ_{22}	φ_{23}
1	$\to$	1	1	1
2	$\to$	2	2	2
3	$\to$	3	3	3
4	$\to$	2	3	1
5	$\to$	0	0	0
6	$\to$	0	0	0 .

Write $z_1\varphi_{2i}$, $i \in \{1,2,3\}$ in terms of basic commutators. Lemma (4.2) implies that those of weight 5 have trivial coefficients. This gives rise to a set of linear equations in the coefficients of the commutators in $v_5^{(1'234)}$. We solve this and obtain that the commutators in $v_5^{(1'234)}$ have trivial coefficients (see (A.4.1.7.), p.A. 63-A.64).

We may now assume, by Lemmas (1.2.11) and (1.2.12), that v is homogeneous of weight 6 and in precisely the generators $1,2,\dots,6$. Write v in the form:

$$v = v_1 + v_2 + v_3,$$

where v_1 is the component of v consisting of left-normed commutators alone, and v_2, v_3 are the components of v consisting respectively of commutators of types $(3,3)$ and $(4,2)$ alone. We map v by the homomorphisms θ_j, $j = 2,3,\dots,6$, where $\theta_j : 1 \to 1,\ 2 \to 1,\dots,j \to 2,\ j+1 \to 1,\dots,6 \to 1$, and obtain immediately that the commutators in v_1 have trivial coefficients. As in the case of the free groups of ranks 3 and 4, it follows from Lemma (3.0.1) that v_2 (and so v_3) is in $\overline{W}^{6}_{6,2}$. Accordingly we deal with v_2 and v_3 separately. Now, since some commutators of types $(3,3)$ and $(4,2)$ have been eliminated (see (A.4.1.8) - (A.4.1.12.),p.A.64-A.72) we may assume that v_2 is a linear combination with integral coefficients of the commutators d_{485}, d_{486}, d_{493}, d_{494}, d_{507} and that v_3 is a linear combination of the commutators d_{530}, d_{531}, d_{532}, d_{533}, d_{534}, d_{535}, d_{540}, d_{541}, d_{550}.

Map v_2 by the homomorphisms:

	φ_{31}	φ_{32}	φ_{33}	φ_{34}	φ_{35}	φ_{36}
$1 \to$	1	1	1	1	1	1
$2 \to$	1	1	1	2	2	2
$3 \to$	1	2	2	1	1	2
$4 \to$	2	1	2	1	2	1
$5 \to$	2	2	1	2	1	1
$6 \to$	2	2	2	2	2	2

The commutators in v_2 have trivial coefficients (see A.4.1.13), p.A.73).

Map v_3 by the homomorphisms:

	φ_{41}	φ_{42}	φ_{43}	φ_{44}	φ_{45}	φ_{46}	φ_{47}	φ_{48}	φ_{49}
$1 \to$	1	1	1	1	1	1	1	1	1
$2 \to$	1	1	1	1	1	1	2	2	2
$3 \to$	1	1	1	2	2	2	1	1	1
$4 \to$	1	2	2	1	1	2	1	1	2
$5 \to$	2	1	2	1	2	1	1	2	1
$6 \to$	2	2	1	2	1	1	2	1	1 .

We note that under φ_{41}, $d_{550} = [6,2,3,4;5,1]$ is the only commutator in v_3 which is mapped into the commutator $[2,1,1,1;2,1]$ and so $e_{550} = 0$. Similarly we can quickly see that e_{530}, e_{531}, e_{532}, 4_{533}, e_{534}, e_{535}, e_{540}, e_{541} are all trivial so that all coefficients in v_3 are trivial.

Hence we conclude that all commutators in v have trivial coefficients, that is v is trivial and we have:

(4.3) <u>Theorem</u>. The set of laws $\{w_1(2,6), w_2(2,6), \ldots, w_7(2,6), [x_1, x_2, \ldots, x_7]\}$ is a basis of laws for $F_2(\underline{\underline{N}}_6)$.

CHAPTER 5

CONCLUSION

We sum up the results which we obtained in Chapters 2 to 4 in order to justify finally the assertions implied by the sketch of the lattice formed by these varieties.

The sets of laws which, together with the appropriate nilpotency law, form a basis for the relevant groups $F_k(\underline{\underline{N}}_m)$ are listed below:

$F_3(\underline{\underline{N}}_5)$:

$$w(3,5) = -[x_4,x_1,x_5;x_3,x_2] + [x_4,x_2,x_5;x_3,x_1] - [x_4,x_3,x_5;x_2,x_1] + [x_3,x_1,x_5;x_4,x_2] - [x_3,x_2,x_5;x_4,x_1] - [x_2,x_1,x_5;x_4,x_3].$$

$F_2(\underline{\underline{N}}_5)$:

$$w_1(2,5) = [x_2,x_1,x_5;x_4,x_3] - [x_4,x_3,x_5;x_2,x_1],$$

$$w_2(2,5) = [x_2,x_1,x_5;x_4,x_3] + [x_3,x_2,x_5;x_4,x_1] - [x_3,x_1,x_5;x_4,x_2].$$

$F_4(\underline{\underline{N}}_6)$:

$$\begin{aligned} w_1(4,6) = {} & -2[x_6,x_5;x_2,x_1;x_4,x_3] + 2[x_6,x_5;x_3,x_1;x_4,x_2] \\ & -2[x_6,x_5;x_3,x_2;x_4,x_1] + [x_4,x_3;x_2,x_1;x_6,x_5] - [x_4,x_2;x_3,x_1;x_6,x_5] \\ & + [x_4,x_1;x_3,x_2;x_6,x_5] - [x_5,x_4;x_3,x_2;x_6,x_1] + [x_5,x_3;x_4,x_2;x_6,x_1] \\ & - [x_5,x_2;x_4,x_3,x_6,x_1] + [x_5,x_4;x_3,x_1,x_6,x_2] - [x_5,x_3;x_4,x_1;x_6,x_2] \\ & + [x_5,x_1;x_4,x_3;x_6,x_2] - [x_5,x_4;x_2,x_1;x_6,x_3] + [x_5,x_2;x_4,x_1;x_6,x_3] \\ & - [x_5,x_1;x_4,x_2;x_6,x_3] + [x_5,x_3;x_2,x_1;x_6,x_4] - [x_5,x_2;x_3,x_1;x_6,x_4] \\ & + [x_5,x_1;x_3,x_2;x_6,x_4], \end{aligned}$$

$$w_2(4,6) = -[x_6,x_4;x_2,x_1;x_5,x_3] + [x_6,x_4;x_3,x_1;x_5,x_2] - [x_6,x_4;x_3,x_2;x_5,x_1]$$
$$+[x_6,x_5;x_2,x_1;x_4,x_3] - [x_6,x_5;x_3,x_1;x_4,x_2] + [x_6,x_5;x_3,x_2;x_4,x_1]$$
$$+[x_5,x_4;x_3,x_2;x_6,x_1] - [x_5,x_4;x_3,x_1;x_6,x_2] + [x_5,x_4;x_2,x_1;x_6,x_3].$$

$F_3(\underline{\underline{N}}_6)$:

$$w_1(3,6) = [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_3,x_5;x_2,x_1,x_6] - [x_4,x_2,x_6;x_3,x_1,x_5]$$
$$+[x_4,x_2,x_5;x_3,x_1,x_6] + [x_4,x_1,x_6;x_3,x_2,x_5] - [x_4,x_1,x_5;x_3,x_2,x_6],$$

$$w_2(3,6) = [x_4,x_3,x_5,x_6;x_2,x_1] - [x_4,x_2,x_5,x_6;x_3,x_1] + [x_4,x_1,x_5,x_6;x_3,x_2]$$
$$+[x_3,x_2,x_5,x_6;x_4,x_1] - [x_3,x_1,x_5,x_6;x_4,x_2] + [x_2,x_1,x_5,x_6;x_4,x_3],$$

$$w_3(3,6) = 2[x_6,x_5;x_2,x_1;x_4,x_3] - 2[x_6,x_5;x_3,x_1;x_4,x_2]$$
$$+2[x_6,x_5;x_3,x_2;x_4,x_1] - [x_4,x_3;x_2,x_1;x_6,x_5]$$
$$+[x_4,x_2;x_3,x_1;x_6,x_5] - [x_4,x_1;x_3,x_2;x_6,x_5],$$

$$w_4(3,6) = -[x_5,x_4;x_3,x_2;x_6,x_1] + [x_5,x_3;x_4,x_2;x_6,x_1]$$
$$-[x_5,x_2;x_4,x_3;x_6,x_1] + [x_5,x_4;x_3,x_1;x_6,x_2]$$
$$-[x_5,x_3;x_4,x_1;x_6,x_2] + [x_5,x_1;x_4,x_3;x_6,x_2]$$
$$-[x_5,x_4;x_2,x_1;x_6,x_3] + [x_5,x_2;x_4,x_1;x_6,x_3]$$
$$-[x_5,x_1;x_4,x_2;x_6,x_3] + [x_5,x_3;x_2,x_1;x_6,x_4]$$
$$-[x_5,x_2;x_3,x_1;x_6,x_4] + [x_5,x_1;x_3,x_2;x_6,x_4],$$

$w_5(3,6) = w_2(4,6)$ (see under $F_4(\underline{\underline{N}}_6)$)

$F_2(\underline{\underline{N}}_6)$:

$$w_1(2,6) = [x_4,x_3,x_5;x_2,x_1,x_6] + [x_4,x_3,x_6;x_2,x_1,x_5],$$

$$w_2(2,6) = [x_2,x_1,x_5,x_6;x_4,x_3] - [x_4,x_3,x_5,x_6;x_2,x_1],$$

$$w_3(2,6) = [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_2,x_6;x_3,x_1,x_5] + [x_4,x_1,x_6;x_3,x_2,x_5],$$

$$w_4(2,6) = -[x_3,x_2,x_5,x_6;x_4,x_1] + [x_3,x_1,x_5,x_6;x_4,x_2]$$
$$-[x_2,x_1,x_5,x_6;x_4,x_3],$$

$w_5(2,6) = [x_1,x_2;x_3,x_4;x_5,x_6]$,

$w_6(2,6) = -[x_6,x_4,x_5;x_2,x_1,x_3] + [x_5,x_3,x_6;x_2,x_1,x_4]$

$w_7(2,6) = w(3,5)$ (see under $F_3(\underline{\underline{N}}_5)$).

The papers [1] and [2], by L.G. Kovács, M.F. Newman, P.F. Pentony and Frank Levin respectively, show that $\underline{\underline{N}}_m = \operatorname{var} F_{m-1}(\underline{\underline{N}}_m)$ for $m > 3$ as well as the fact that $\operatorname{var} F_{m-2}(\underline{\underline{N}}_m) \subset \underline{\underline{N}}_m$. That $\operatorname{var} F_2(\underline{\underline{N}}_6) \subset \operatorname{var} F_3(\underline{\underline{N}}_6) \subset \operatorname{var} F_4(\underline{\underline{N}}_6)$ and $\operatorname{var} F_2(\underline{\underline{N}}_5) \subset \operatorname{var} F_3(\underline{\underline{N}}_5)$ has been established in [2], but it clearly is also an immediate consequence of the bases of laws for these groups that we provide and their derivation in the relevant chapters.

To establish that $\underline{\underline{N}}_5 \wedge \operatorname{var} F_2(\underline{\underline{N}}_6) = \operatorname{var} F_3(\underline{\underline{N}}_5)$ notice that the laws of the left hand side are the union of the laws of $\underline{\underline{N}}_5$ and those of $F_2(\underline{\underline{N}}_6)$, and these are satisfied in $F_3(\underline{\underline{N}}_5)$. Thus $\underline{\underline{N}}_5 \wedge \operatorname{var} F_2(\underline{\underline{N}}_6) \supseteq \operatorname{var} F_3(\underline{\underline{N}}_5)$. On the other hand every group in the left hand side is of class 5 and satisfies the only other law, $w(3,5)$, of $F_3(\underline{\underline{N}}_5)$, hence belongs to $\operatorname{var} F_3(\underline{\underline{N}}_5)$, giving the reverse inclusion, and hence the result.

To prove that $\underline{\underline{N}}_5 \vee \operatorname{var} F_2(\underline{\underline{N}}_6) \subset \operatorname{var} F_3(\underline{\underline{N}}_6)$ notice that $\underline{\underline{N}}_5 \subset \operatorname{var} F_3(\underline{\underline{N}}_6)$ because the latter has no laws of weight 5; as $\operatorname{var} F_2(\underline{\underline{N}}_6) \subset \operatorname{var} F_3(\underline{\underline{N}}_6)$, we certainly know that the union of the two varieties is contained in $\operatorname{var} F_3(\underline{\underline{N}}_6)$. To confirm that the inclusion is proper, notice that, for example, the law $[x_1,x_2;x_3,x_4;x_5,x_6]$ is satisfied in the variety on the left hand side, but not in $F_3(\underline{\underline{N}}_6)$.

Finally, $\underline{\underline{N}}_4$ is contained in var $F_2(\underline{\underline{N}}_5)$, because any group that is nilpotent of class 4 satisfies all the laws of $N_2(\underline{\underline{N}}_5)$ since all of these are of weight five; that the inclusion is proper, is trivial.

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APPENDICES

APPENDIX

Appendix 1: A list of the relevant basic commutators.

N.B. notational convention used in these tables ONLY: The variable x_i will be replaced by the figure i for easier reading. That is for [2,1,1,1] say, read $[x_2,x_1,x_1,x_1]$.

Weight 4 in 2 variables.

$c_1 = [2,1,1,1]$, $c_2 = [2,1,1,2]$, $c_3 = [2,1,2,2]$.

Weight 4 in 3 variables.

$c_4 = [2,1,1,3]$, $c_5 = [2,1,2,3]$, $c_6 = [2,1,3,3]$,

$c_7 = [3,1,1,2]$, $c_8 = [3,1,2,2]$, $c_9 = [3,1,2,3]$,

$c_{10} = [3,1;2,1]$, $c_{11} = [3,2;2,1]$, $c_{12} = [3,2;3,1]$.

Weight 4 in 4 variables.

$c_{13} = [2,1,3,4]$, $c_{14} = [3,1,2,4]$, $c_{15} = [4,1,2,3]$,

$c_{16} = [4,3;2,1]$, $c_{17} = [4,2;3,1]$, $c_{18} = [4,1;3,2]$.

Weight 5 in 2 variables.

$c_{19} = [2,1,1,1,1]$, $c_{20} = [2,1,1,1,2]$, $c_{21} = [2,1,1,2,2]$,

$c_{22} = [2,1,2,2,2]$,

$c_{23} = [2,1,1;2,1]$, $c_{24} = [2,1,2;2,1]$.

Weight 5 in 3 variables

$c_{25} = [2,1,1,1,3]$, $c_{26} = [2,1,1,2,3]$, $c_{27} = [2,1,1,3,3]$,

$c_{28} = [2,1,2,2,3]$, $c_{29} = [2,1,2,3,3]$,

$c_{30} = [2,1,3,3,3]$, $c_{31} = [3,1,1,1,2]$, $c_{32} = [3,1,1,2,2]$,

$c_{33} = [3,1,1,2,3]$, $c_{34} = [3,1,2,2,2]$, $c_{35} = [3,1,2,2,3]$,

$c_{36} = [3,1,2,3,3]$,

$c_{37} = [2,1,3;2,1]$, $c_{38} = [3,1,1;2,1]$, $c_{39} = [3,1,2;2,1]$,

$c_{40} = [3,1,3;2,1]$, $c_{41} = [3,2,2;2,1]$, $c_{42} = [3,2,3;2,1]$,

$c_{43} = [2,1,1;3,1]$, $c_{44} = [2,1,2;3,1]$, $c_{45} = [2,1,3;3,1]$,

$c_{46} = [3,1,2;3,1]$, $c_{47} = [3,2,2;3,1]$, $c_{48} = [3,2,3;3,1]$,

$c_{49} = [2,1,1;3,2]$, $c_{50} = [2,1,2;3,2]$, $c_{51} = [2,1,3;3,2]$,

$c_{52} = 3,1,1;3,2]$, $c_{53} = [3,1,2;3,2]$, $c_{54} = [3,1,3;3,2]$.

Weight 5 in 4 variables.

$c_{55} = [2,1,1,3,4]$, $c_{56} = [2,1,2,3,4]$, $c_{57} = [2,1,3,3,4]$,

$c_{58} = [2,1,3,4,4]$, $c_{59} = [3,1,1,2,4]$, $c_{60} = [3,1,2,2,4]$,

$c_{61} = [3,1,2,3,4]$, $c_{62} = [3,1,2,4,4]$, $c_{63} = [4,1,1,2,3]$,

$c_{64} = [4,1,2, ,3]$, $c_{65} = [4,1,2,2,3]$, $c_{66} = [4,1,2,3,4]$,

$c_{67} = [3,1,4;2,1]$, $c_{68} = [3,2,4;2,1]$, $c_{69} = [4,1,3;2,1]$,

$c_{70} = [4,2,3;2,1]$, $c_{71} = [4,3,3;2,1]$, $c_{72} = [4,3,4;2,1]$,

$c_{73} = [2,1,4;3,1]$, $c_{74} = [3,2,4;3,1]$, $c_{75} = [4,1,2;3,1]$,

$c_{76} = [4,2,2;3,1]$, $c_{77} = [4,2,3;3,1]$, $c_{78} = [4,2,4;3,1]$,

$c_{79} = [2,1,4;3,2]$, $c_{80} = [3,1,4;3,2]$, $c_{81} = [4,1,1;3,2]$,

$c_{82} = [4,1,2;3,2]$, $c_{83} = [4,1,3;3,2]$, $c_{84} = [4,1,4;3,2]$,

$c_{85} = [2,1,3;4,1]$, $c_{86} = [3,1,2;4,1]$, $c_{87} = [3,2,2;4,1]$,

$c_{88} = [3,2,3;4,1]$, $c_{89} = [3,2,4;4,1]$, $c_{90} = [4,2,3;4,1]$,

$c_{91} = [2,1,3;4,2]$, $c_{92} = [3,1,1;4,2]$, $c_{93} = [3,1,2;4,2]$,

$c_{94} = [3,1,3;4,2]$, $c_{95} = [3,1,4;4,2]$, $c_{96} = [4,1,3;4,2]$,

$c_{97} = [2,1,1;4,3]$, $c_{98} = [2,1,2;4,3]$, $c_{99} = [2,1,3;4,3]$,

$c_{100} = [2,1,4;4,3]$, $c_{101} = [3,1,2;4,3]$, $c_{102} = [4,1,2;4,3]$.

Weight 5 in 5 variables.

$c_{103} = [2,1,3,4,5]$, $c_{104} = [3,1,2,4,5]$, $c_{105} = [4,1,2,3,5]$,

$c_{106} = [5,1,2,3,4]$,

$c_{107} = [4,3,5;2,1]$, $c_{108} = [5,3,4;2,1]$, $c_{109} = [4,2,5;3,1]$,

$c_{110} = [5,2,4;3,1]$, $c_{111} = [4,1,5;3,2]$, $c_{112} = [5,1,4;3,2]$,

$c_{113} = [3,2,5;4,1]$, $c_{114} = [5,2,3;4,1]$, $c_{115} = [3,1,5;4,2]$,

$c_{116} = [5,1,3;4,2]$, $c_{117} = [2,1,5;4,3]$, $c_{118} = [5,1,2;4,3]$,

$c_{119} = [3,2,4;5,1]$, $c_{120} = [4,2,3;5,1]$, $c_{121} = [3,1,4;5,2]$,

$c_{122} = [4,1,3;5,2]$, $c_{123} = [2,1,4;5,3]$, $c_{124} = [4,1,2;5,3]$,

$c_{125} = [2,1,3;5,4]$, $c_{126} = [3,1,2;5,4]$.

Weight 6 in 2 variables.

$c_{127} = [2,1,1,1,1,1]$, $c_{128} = [2,1,1,1,1,2]$, $c_{129} = [2,1,1,1,2,2]$,

$c_{130} = [2,1,1,2,2,2]$, $c_{131} = [2,1,2,2,2,2]$,

$c_{132} = [2,1,2;2,1,1]$, $c_{133} = [2,1,1,1;2,1]$, $c_{134} = [2,1,1,2;2,1]$,

$c_{135} = [2,1,2,2;2,1]$.

Weight 6 in 3 variables.

$c_{136} = [2,1,1,1,1,3]$, $c_{137} = [2,1,1,1,2,3]$, $c_{138} = [2,1,1,1,3,3]$,
$c_{139} = [2,1,1,2,2,3]$, $c_{140} = [2,1,1,2,3,3]$, $c_{141} = [2,1,1,3,3,3]$,
$c_{142} = [2,1,2,2,2,3]$, $c_{143} = [2,1,2,2,3,3]$, $c_{144} = [2,1,2,3,3,3]$,
$c_{145} = [2,1,3,3,3,3]$, $c_{146} = [3,1,1,1,1,2]$, $c_{147} = [3,1,1,1,2,2]$,
$c_{148} = [3,1,1,1,2,3]$, $c_{149} = [3,1,1,2,2,2]$, $c_{150} = [3,1,1,2,2,3]$,
$c_{151} = [3,1,1,2,3,3]$, $c_{152} = [3,1,2,2,2,2]$, $c_{153} = [3,1,2,2,2,3]$,
$c_{154} = [3,1,2,2,3,3]$, $c_{155} = [3,1,2,3,3,3]$,

$c_{156} = [2,1,3;2,1,1]$, $c_{157} = [3,1,1;2,1,1]$, $c_{158} = [3,1,2;2,1,1]$,
$c_{159} = [3,1,3;2,1,1]$, $c_{160} = [3,2,2;2,1,1]$, $c_{161} = [3,2,3;2,1,1]$,
$c_{162} = [2,1,3;2,1,2]$, $c_{163} = [3,1,1;2,1,2]$, $c_{164} = [3,1,2;2,1,2]$,
$c_{165} = [3,1,3;2,1,2]$, $c_{166} = [3,2,2;2,1,2]$, $c_{167} = [3,2,2;2,1,2]$,
$c_{168} = [3,1,1\ ;2,1,3]$, $c_{169} = [3,1,2;2,1,3]$, $c_{170} = [3,1,3;2,1,3]$,
$c_{171} = [3,2,2;2,1,3]$, $c_{172} = [3,2,2;2,1,3]$, $c_{173} = [3,1,2;3,1,1]$,
$c_{174} = [3,2,2;3,1,1]$, $c_{175} = [3,2,3;3,1,1]$, $c_{176} = [3,1,3;3,1,2]$,
$c_{177} = [3,2,2;3,1,2]$, $c_{178} = [3,2,3;3,1,2]$, $c_{179} = [3,2,2;3,1,3]$,
$c_{180} = [3,2,3;3,1,3]$,

$c_{181} = [2,1,1,3;2,1]$, $c_{182} = [2,1,2,3;2,1]$, $c_{183} = [2,1,3,3;2,1]$,
$c_{184} = [3,1,1,1;2,1]$, $c_{185} = [3,1,1,2;2,1]$, $c_{186} = [3,1,1,3;2,1]$,
$c_{187} = [3,1,2,2;2,1]$, $c_{188} = [3,1,2,3;2,1]$, $c_{189} = [3,1,3,3;2,1]$,
$c_{190} = [3,2,2,2;2,1]$, $c_{191} = [3,2,2,3;2,1]$, $c_{192} = [3,2,3,3;2,1]$,
$c_{193} = [2,1,1,1;3,1]$, $c_{194} = [2,1,1,2;3,1]$, $c_{195} = [2,1,1,3;3,1]$,
$c_{196} = [2,1,2,2;3,1]$, $c_{197} = [2,1,2,3;3,1]$, $c_{198} = [2,1,3,3;3,1]$,
$c_{199} = [3,1,1,2;3,1]$, $c_{200} = [3,1,2,2;3,1]$, $c_{201} = [3,1,2,3;3,1]$,

$c_{202} = [3,2,2,2;3,1]$, $c_{203} = [3,2,2,3;3,1]$, $c_{204} = [3,2,3,3;3,1]$,
$c_{205} = [2,1,1,1;3,2]$, $c_{206} = [2,1,1,2;3,2]$, $c_{207} = [2,1,1,3;3,2]$,
$c_{208} = [2,1,2,2;3,2]$, $c_{209} = [2,1,2,3;3,2]$, $c_{210} = [2,1,3,3;3,2]$,
$c_{211} = [3,1,1,1;3,2]$, $c_{212} = [3,1,1,2;3,2]$, $c_{213} = [3,1,1,3;3,2]$,
$c_{214} = [3,1,2,2;3,2]$, $c_{215} = [3,1,2,3;3,2]$, $c_{216} = [3,1,3,3;3,2]$,
$c_{217} = [3,1;2,1;2,1]$, $c_{218} = [3,2;2,1;2,1]$, $c_{219} = [3,1;2,1;3,1]$,
$c_{220} = [3,2;2,1;3,1]$, $c_{221} = [3,2;3,1;3,1]$, $c_{222} = [3,1;2,1;3,2]$,
$c_{223} = [3,2;2,1;3,2]$, $c_{224} = [3,2;3,1;3,2]$.

<u>Weight 6 in 4 variables.</u>

$c_{225} = [2,1,1,1,3,4]$, $c_{226} = [2,1,1,2,3,4]$, $c_{227} = [2,1,1,3,3,4]$,
$c_{228} = [2,1,1,3,4,4]$, $c_{229} = [2,1,2,2,3,4]$, $c_{230} = [2,1,2,3,3,4]$,
$c_{231} = [2,1,2,3,4,4]$, $c_{232} = [2,1,3,3,3,4]$, $c_{233} = [2,1,3,3,4,4]$,
$c_{234} = [2,1,3,4,4,4]$, $c_{235} = [3,1,1,1,2,4]$, $c_{236} = [3,1,1,2,2,4]$,
$c_{237} = [3,1,1,2,3,4]$, $c_{238} = [3,1,1,2,4,4]$, $c_{239} = [3,1,2,2,2,4]$,
$c_{240} = [3,1,2,2,3,4]$, $c_{241} = [3,1,2,2,4,4]$, $c_{242} = [3,1,2,3,3,4]$,
$c_{243} = [3,1,2,3,4,4]$, $c_{244} = [3,1,2,4,4,4]$, $c_{245} = [4,1,1,1,2,3]$,
$c_{246} = [4,1,1,2,2,3]$, $c_{247} = [4,1,1,2,3,3]$, $c_{248} = [4,1,1,2,3,4]$,
$c_{249} = [4,1,2,2,2,3]$, $c_{250} = [4,1,2,2,3,3,]$, $c_{251} = [4,1,2,2,3,4]$,
$c_{252} = [4,1,2,3,3,3]$, $c_{253} = [4,1,2,3,3,4]$, $c_{254} = [4,1,2,3,4,4]$,
$c_{255} = [3,1,4;2,1,1]$, $c_{256} = [3,2,4;2,1,1]$, $c_{257} = [4,1,3;2,1,1]$,
$c_{258} = [4,2,3;2,1,1]$, $c_{259} = [4,3,3;2,1,1]$, $c_{260} = [4,3,4;2,1,1]$,
$c_{261} = [3,1,4;2,1,2]$, $c_{262} = [3,2,4;2,1,2]$, $c_{263} = [4,1,3;2,1,2]$,
$c_{264} = [4,2,3;2,1,2]$, $c_{265} = [4,3,3;2,1,2]$, $c_{266} = [4,3,4;2,1,2]$,

$c_{267} = [2,1,4;2,1,3]$, $c_{268} = [3,1,4;2,1,3]$, $c_{269} = [3,2,4;2,1,3]$,

$c_{270} = [4,1,1;2,1,3]$, $c_{271} = [4,1,2;2,1,3]$, $c_{272} = [4,1,3;2,1,3]$,

$c_{273} = [4,1,4;2,1,3]$, $c_{274} = [4,2,2;2,1,3]$, $c_{275} = [4,2,3;2,1,3]$,

$c_{276} = [4,2,4;2,1,3]$, $c_{277} = [4,3,3;2,1,3]$, $c_{278} = [4,3,4;2,1,3]$,

$c_{279} = [3,1,1;2,1,4]$, $c_{280} = [3,1,2;2,1,4]$, $c_{281} = [3,1,3;2,1,4]$,

$c_{282} = [3,1,4;2,1,4]$, $c_{283} = [3,2,2;2,1,4]$, $c_{284} = [3,2,3;2,1,4]$,

$c_{285} = [3,2,4;2,1,4]$, $c_{286} = [4,1,3;2,1,4]$, $c_{287} = [4,2,3;2,1,4]$,

$c_{288} = [4,3,3;2,1,4]$, $c_{289} = [4,3,4;2,1,4]$, $c_{290} = [3,2,4;3,1,1]$,

$c_{291} = [4,1,2;3,1,1]$, $c_{292} = [4,2,2;3,1,1]$, $c_{293} = [4,2,3;3,1,1]$,

$c_{294} = [4,2,4;3,1,1]$, $c_{295} = [3,1,4;3,1,2]$, $c_{296} = [3,2,4;3,1,2]$,

$c_{297} = [4,1,1;3,1,2]$, $c_{298} = [4,1,2;3,1,2]$, $c_{299} = [4,1,3;3,1,2]$,

$c_{300} = [4,1,4;3,1,2]$, $c_{301} = [4,2,2;3,1,2]$, $c_{302} = [4,2,3;3,1,2]$,

$c_{303} = [4,2,4;3,1,2]$, $c_{304} = [4,3,3;3,1,2]$, $c_{305} = [4,3,4;3,1,2]$,

$c_{306} = [3,2,4;3,1,3]$, $c_{307} = [4,1,2;3,1,3]$, $c_{308} = [4,2,2;3,1,3]$,

$c_{309} = [4,2,3;3,1,3]$, $c_{310} = [4,2,4;3,1,3]$, $c_{311} = [3,2,2;3,1,4]$,

$c_{312} = [3,2,3;3,1,4]$, $c_{313} = [3,2,4;3,1,4]$, $c_{314} = [4,1,2;3,1,4]$,

$c_{315} = [4,2,2;3,1,4]$, $c_{316} = [4,2,3;3,1,4]$, $c_{317} = [4,2,4;3,1,4]$,

$c_{318} = [4,1,1;3,2,2]$, $c_{319} = [4,1,2;3,2,2]$, $c_{320} = [4,1,3;3,2,2]$,

$c_{321} = [4,1,4;3,2,2]$, $c_{322} = [4,1,1;3,2,3]$, $c_{323} = [4,1,2;3,2,3]$,

$c_{324} = [4,1,3;3,2,3]$, $c_{325} = [4,1,4;3,2,3]$, $c_{326} = [4,1,1;3,2,4]$,

$c_{327} = [4,1,2;3,2,4]$, $c_{328} = [4,1,3;3,2,4]$, $c_{329} = [4,1,4;3,2,4]$,

$c_{330} = [4,2,3;4,1,1]$, $c_{331} = [4,1,3;4,1,2]$, $c_{332} = [4,2,3;4,1,2]$,

$c_{333} = [4,3,3;4,1,2]$, $c_{334} = [4,3,4;4,1,2]$, $c_{335} = [4,2,2;4,1,3]$,

$c_{336} = [4,2,3;4,1,3]$, $c_{337} = [4,2,4;4,1,3]$, $c_{338} = [4,2,3;4,1,4]$,

$c_{339} = [2,1,3,4;2,1]$, $c_{340} = [3,1,1,4;2,1]$, $c_{341} = [3,1,2,4;2,1]$,

$c_{342} = [3,1,3,4;2,1]$, $c_{343} = [3,1,4,4;2,1]$, $c_{344} = [3,2,2,4;2,1]$,

$c_{345} = [3,2,3,4;2,1]$, $c_{346} = [3,2,4,4;2,1]$, $c_{347} = [4,1,1,3;2,1]$,

$c_{348} = [4,1,2,3;2,1]$, $c_{349} = [4,1,3,3;2,1]$, $c_{350} = [4,1,3,4;2,1]$,

$c_{351} = [4,2,2,3;2,1]$, $c_{352} = [4,2,3,3;2,1]$, $c_{353} = [4,2,3,4;2,1]$,

$c_{354} = [4,3,3,3;2,1]$, $c_{355} = [4,3,3,4;2,1]$, $c_{356} = [4,3,4,4;2,1]$,

$c_{357} = [2,1,1,4;3,1]$, $c_{358} = [2,1,2,4;3,1]$, $c_{359} = [2,1,3,4;3,1]$,

$c_{360} = [2,1,4,4;3,1]$, $c_{361} = [3,1,2,4;3,1]$, $c_{362} = [3,2,2,4;3,1]$,

$c_{363} = [3,2,3,4;3,1]$, $c_{364} = [3,2,4,4;3,1]$, $c_{365} = [4,1,1,2;3,1]$,

$c_{366} = [4,1,2,2;3,1]$, $c_{367} = [4,1,2,3;3,1]$, $c_{368} = [4,1,2,4;3,1]$,

$c_{369} = [4,2,2,2;3,1]$, $c_{370} = [4,2,2,3;3,1]$, $c_{371} = [4,2,2,4;3,1]$,

$c_{372} = [4,2,3,3;3,1]$, $c_{373} = [4,2,3,4;3,1]$, $c_{374} = [4,2,4,4;3,1]$,

$c_{375} = [2,1,1,4;3,2]$, $c_{376} = [2,1,2,4;3,2]$, $c_{377} = [2,1,3,4;3,2]$,

$c_{378} = [2,1,4,4;3,2]$, $c_{379} = [3,1,1,4;3,2]$, $c_{380} = [3,1,2,4;3,2]$,

$c_{381} = [3,1,3,4;3,2]$, $c_{382} = [3,1,4,4;3,2]$, $c_{383} = [4,1,1,1;3,2]$,

$c_{384} = [4,1,1,2;3,2]$, $c_{385} = [4,1,1,3;3,2]$, $c_{386} = [4,1,1,4;3,2]$,

$c_{387} = [4,1,2,2;3,2]$, $c_{388} = [4,1,2,3;3,2]$, $c_{389} = [4,1,2,4;3,2]$,

$c_{390} = [4,1,3,3;3,2]$, $c_{391} = [4,1,3,4;3,2]$, $c_{392} = [4,1,4,4;3,2]$,

$c_{393} = [2,1,1,3;4,1]$, $c_{394} = [2,1,2,3;4,1]$, $c_{395} = [2,1,3,3;4,1]$,

$c_{396} = [2,1,3,4;4,1]$, $c_{397} = [3,1,1,2;4,1]$, $c_{398} = [3,1,2,2;4,1]$,

$c_{399} = [3,1,2,3;4,1]$, $c_{400} = [3,1,2,4;4,1]$, $c_{401} = [3,2,2,2;4,1]$,

$c_{402} = [3,2,2,3;4,1]$, $c_{403} = [3,2,2,4;4,1]$, $c_{404} = [3,2,3,3;4,1]$,

$c_{405} = [3,2,3,4;4,1]$, $c_{406} = [3,2,4,4;4,1]$, $c_{407} = [4,1,2,3;4,1]$,

$c_{408} = [4,2,2,3;4,1]$, $c_{409} = [4,2,3,3;4,1]$, $c_{410} = [4,2,3,4;4,1]$,

$c_{411} = [2,1,1,3;4,2]$, $c_{412} = [2,1,2,3;4,2]$, $c_{413} = [2,1,3,3;4,2]$,

$c_{414} = [2,1,3,4;4,2]$, $c_{415} = [3,1,1,1;4,2]$, $c_{416} = [3,1,1,2;4,2]$,

$c_{417} = [3,1,1,3;4,2]$, $c_{418} = [3,1,1,4;4,2]$, $c_{419} = [3,1,2,2;4,2]$,

$c_{420} = [3,1,2,3;4,2]$, $c_{421} = [3,1,2,4;4,2]$, $c_{422} = [3,1,3,3;4,2]$,

$c_{423} = [3,1,3,4;4,2]$, $c_{424} = [3,1,4,4;4,2]$, $c_{425} = [4,1,1,3;4,2]$,

$c_{426} = [4,1,2,3;4,2]$, $c_{427} = [4,1,3,3;4,2]$, $c_{428} = [4,1,3,4;4,2]$,

$c_{429} = [2,1,1,1;4,3]$, $c_{430} = [2,1,1,2;4,3]$, $c_{431} = [2,1,1,3;4,3]$,

$c_{432} = [2,1,1,4;4,3]$, $c_{433} = [2,1,2,2;4,3]$, $c_{434} = [2,1,2,3;4,3]$,

$c_{435} = [2,1,2,4;4,3]$, $c_{436} = [2,1,3,3;4,3]$, $c_{437} = [2,1,3,4;4,3]$,

$c_{438} = [2,1,4,4;4,3]$, $c_{439} = [3,1,1,2;4,3]$, $c_{440} = [3,1,2,2;4,3]$,

$c_{441} = [3,1,2,3;4,3]$, $c_{442} = [3,1,2,4;4,3]$, $c_{443} = [4,1,1,2;4,3]$,

$c_{444} = [4,1,2,2;4,3]$, $c_{445} = [4,1,2,3;4,3]$, $c_{446} = [4,1,2,4;4,3]$,

$c_{447} = [4,3;2,1;2,1]$, $c_{448} = [4,1;2,1;3,1]$, $c_{449} = [4,2;2,1;3,1]$,

$c_{450} = [4,3;2,1;3,1]$, $c_{451} = [4,2;3,1;3,1]$, $c_{452} = [4,1;2,1;3,2]$,

$c_{453} = [4,2;2,1;3,2]$, $c_{454} = [4,3;2,1;3,2]$, $c_{455} = [4,1;3,1;3,2]$,

$c_{456} = [4,2;3,1;3,2]$, $c_{457} = [4,3;3,1;3,2]$, $c_{458} = [4,1;3,2;3,2]$,

$c_{459} = [3,1;2,1;4,1]$, $c_{460} = [3,2;2,1;4,1]$, $c_{461} = [4,3;2,1;4,1]$,

$c_{462} = [3,2;3,1;4,1]$, $c_{463} = [4,2;3,1;4,1]$, $c_{464} = [4,1;3,2;4,1]$,

$c_{465} = [4,2;3,2;4,1]$, $c_{466} = [4,3;3,2;4,1]$, $c_{467} = [3,1;2,1;4,2]$,

$c_{468} = [3,2;2,1;4,2]$, $c_{469} = [4,3;2,1;4,2]$, $c_{470} = [3,2;3,1;4,2]$,

$c_{471} = [4,1;3,1;4,2]$, $c_{472} = [4,2;3,1;4,2]$, $c_{473} = [4,3;3,1;4,2]$,

$c_{474} = [4,1;3,2;4,2]$, $c_{475} = [4,3;4,1;4,2]$, $c_{476} = [3,1;2,1;4,3]$,

$c_{477} = [3,2;2,1;4,3]$, $c_{478} = [4,1;2,1;4,3]$, $c_{479} = [4,2;2,1;4,3]$,
$c_{480} = [4,3;2,1;4,3]$, $c_{481} = [3,2;3,1;4,3]$, $c_{482} = [4,2;3,1;4,3]$,
$c_{483} = [4,1;3,2;4,3]$, $c_{484} = [4,2;4,1;4,3]$.

Weight 6 in 6 variables.

$c_{485} = [2,1,3,4,5,6]$, $c_{486} = [3,1,2,4,5,6]$, $c_{487} = [4,1,2,3,5,6]$,
$c_{488} = [5,1,2,3,4,6]$, $c_{489} = [6,1,2,3,4,5]$,
$c_{490} = [5,4,6;2,1,3]$, $c_{491} = [6,4,5;2,1,3]$, $c_{492} = [5,3,6;2,1,4]$,
$c_{493} = [6,3,5;2,1,4]$, $c_{494} = [4,3,6;2,1,5]$, $c_{495} = [6,3,4;2,1,5]$,
$c_{496} = [4,3,5;2,1,6]$, $c_{497} = [5,3,4;2,1,6]$, $c_{498} = [5,4,6;3,1,2]$,
$c_{499} = [6,4,5;3,1,2]$, $c_{500} = [5,2,6;3,1,4]$, $c_{501} = [6,2,5;3,1,4]$,
$c_{502} = [4,2,6;3,1,5]$, $c_{503} = [6,2,4;3,1,5]$, $c_{504} = [4,2,5;3,1,6]$,
$c_{505} = [5,2,4;3,1,6]$, $c_{506} = [5,1,6;3,2,4]$, $c_{507} = [6,1,5;3,2,4]$,
$c_{508} = [4,1,6;3,2,5]$, $c_{509} = [6,1,4;3,2,5]$, $c_{510} = [4,1,5;3,2,6]$,
$c_{511} = [5,1,4;3,2,6]$, $c_{512} = [5,3,6;4,1,2]$, $c_{513} = [6,3,5;4,1,2]$,
$c_{514} = [5,2,6;4,1,3]$, $c_{515} = [6,2,5;4,1,3]$, $c_{516} = [6,2,3;4,1,5]$,
$c_{517} = [5,2,3;4,1,6]$, $c_{518} = [5,1,6;4,2,3]$, $c_{519} = [6,1,5;4,2,3]$,
$c_{520} = [6,1,3;4,2,5]$, $c_{521} = [5,1,3;4,2,6]$, $c_{522} = [6,1,2;4,3,5]$,
$c_{523} = [5,1,2;4,3,6]$, $c_{524} = [6,3,4;5,1,2]$, $c_{525} = [6,2,4;5,1,3]$,
$c_{526} = [6,2,3;5,1,4]$, $c_{527} = [6,1,4;5,2,3]$, $c_{528} = [6,1,3;5,2,4]$,
$c_{529} = [6,1,2;5,3,4]$,
$c_{530} = [4,3,5,6;2,1]$, $c_{531} = [5,3,4,6;2,1]$, $c_{532} = [6,3,4,5;2,1]$,
$c_{533} = [4,2,5,6;3,1]$, $c_{534} = [5,2,4,6;3,1]$, $c_{535} = [6,2,4,5;3,1]$,

$c_{536} = [4,1,5,6;3,2]$, $c_{537} = [5,1,4,6;3,2]$, $c_{538} = [6,1,4,5;3,2]$,

$c_{539} = [3,2,5,6;4,1]$, $c_{540} = [5,2,3,6;4,1]$, $c_{541} = [6,2,3,5;4,1]$,

$c_{542} = [3,1,5,6;4,2]$, $c_{543} = [5,1,3,6;4,2]$, $c_{544} = [6,1,3,5;4,2]$,

$c_{545} = [2,1,5,6;4,3]$, $c_{546} = [5,1,2,6;4,3]$, $c_{547} = [6,1,2,5;4,3]$,

$c_{548} = [3,2,4,6;5,1]$, $c_{549} = [4,2,3,6;5,1]$, $c_{550} = [6,2,3,4;5,1]$,

$c_{551} = [3,1,4,6;5,2]$, $c_{552} = [4,1,3,6;5,2]$, $c_{553} = [6,1,3,4;5,2]$,

$c_{554} = [2,1,4,6;5,3]$, $c_{555} = [4,1,2,6;5,3]$, $c_{556} = [6,1,2,4;5,3]$,

$c_{557} = [2,1,3,6;5,4]$, $c_{558} = [3,1,2,6;5,4]$, $c_{559} = [6,1,2,3;5,4]$,

$c_{560} = [3,2,4,5;6,1]$, $c_{561} = [4,2,3,5;6,1]$, $c_{562} = [5,2,3,4;6,1]$,

$c_{563} = [3,1,4,5;6,2]$, $c_{564} = [4,1,3,5;6,2]$, $c_{565} = [5,1,3,4;6,2]$,

$c_{566} = [2,1,4,5;6,3]$, $c_{567} = [4,1,2,5;6,3]$, $c_{568} = [5,1,2,4;6,3]$,

$c_{569} = [2,1,3,5;6,4]$, $c_{570} = [3,1,2,5;6,4]$, $c_{571} = [5,1,2,3;6,4]$,

$c_{572} = [2,1,3,4;6,5]$, $c_{573} = [3,1,2,4;6,5]$, $c_{574} = [4,1,2,3;6,5]$,

$c_{575} = [6,5;3,2;4,1]$, $c_{576} = [6,5;3,1;4,2]$, $c_{577} = [6,5;2,1;4,3]$,

$c_{578} = [6,4;3,2;5,1]$, $c_{579} = [6,3;4,2;5,1]$, $c_{580} = [6,2;4,3;5,1]$,

$c_{581} = [6,4;3,1;5,2]$, $c_{582} = [6,3;4,1;5,2]$, $c_{583} = [6,1;4,3;5,2]$,

$c_{584} = [6,4;2,1;5,3]$, $c_{585} = [6,2;4,1;5,3]$, $c_{586} = [6,1;4,2;5,3]$,

$c_{587} = [6,3;2,1;5,4]$, $c_{588} = [6,2;3,1;5,4]$, $c_{589} = [6,1;3,2;5,4]$,

$c_{590} = [5,4;3,2;6,1]$, $c_{591} = [5,3;4,2;6,1]$, $c_{592} = [5,2;4,3;6,1]$,

$c_{593} = [5,4;3,1;6,2]$, $c_{594} = [5,3;4,1;6,2]$, $c_{595} = [5,1;4,3;6,2]$,

$c_{596} = [5,4;2,1;6,3]$, $c_{597} = [5,2;4,1;6,3]$, $c_{598} = [5,1;4,2;6,3]$,

$c_{599} = [5,3;2,1;6,4]$, $c_{600} = [5,2;3,1;6,4]$, $c_{601} = [5,1;3,2;6,4]$,

$c_{602} = [4,3;2,1;6,5]$, $c_{603} = [4,2;3,1;6,5]$, $c_{604} = [4,1;3,2;6,5]$.

Appendix 2: Computations for Chapter 2.

(A.2.1.1) The law $w(3,5)$ in $F_3(\underline{\underline{N}}_5)$ given by $\sum_{\sigma\varepsilon S} \varepsilon(\sigma)[x_5,x_{4\sigma},x_{3\sigma},x_{2\sigma},x_{1\sigma}]$ (p.19).

An explicit expression for $w(3,5)$

$$\begin{aligned} w(3,5) = {} & [x_5,x_1,x_2,x_3,x_4] - [x_5,x_2,x_1,x_3,x_4] - [x_5,x_1,x_2,x_4,x_3] + \\ & [x_5,x_2,x_1,x_4,x_3] - [x_5,x_1,x_3,x_2,x_4] + [x_5,x_3,x_1,x_2,x_4] + \\ & [x_5,x_1,x_3,x_4,x_2] - [x_5,x_3,x_1,x_4,x_2] + [x_5,x_1,x_4,x_2,x_3] - \\ & [x_5,x_4,x_1,x_2,x_3] - [x_5,x_1,x_4,x_3,x_2] + [x_5,x_4,x_1,x_3,x_2] + \\ & [x_5,x_2,x_3,x_1,x_4] - [x_5,x_3,x_2,x_1,x_4] - [x_5,x_2,x_3,x_4,x_1] + \\ & [x_5,x_3,x_2,x_4,x_1] - [x_5,x_2,x_4,x_1,x_3] + [x_5,x_4,x_2,x_1,x_3] + \\ & [x_5,x_2,x_4,x_3,x_1] - [x_5,x_4,x_2,x_3,x_1] + [x_5,x_3,x_4,x_1,x_2] - \\ & [x_5,x_4,x_3,x_1,x_2] - [x_5,x_3,x_4,x_2,x_1] + [x_5,x_4,x_3,x_2,x_1] . \end{aligned}$$

$w(3,5)$ in terms of basic commutators.

To express $w(3,5)$ in terms of basic commutators we need the following rather trivial lemma.

Lemma. In a nilpotent group of class 5, the following identities which can be proved by a simple application of the identities listed in 0.2, hold.

(a) $[x,y,z,t,u] = + [z,y,x,t,u] - [z,x,y,t,u]$

(b) $[x,y,z,t,u] - [x,y,z,u,t] = - [x,y,z;u,t]$.

Now identity (a) implies that $w(3,5)$ can be written in the form:

$$w(3,5) = [x_2,x_1,x_5,x_3,x_4] - [x_2,x_1,x_5,x_4,x_3] - [x_3,x_1,x_5,x_2,x_4] + [x_3,x_1,x_5,x_4,x_2] + [x_4,x_1,x_5,x_2,x_3] - [x_4,x_1,x_5,x_3,x_2] + [x_3,x_2,x_5,x_1,x_4] - [x_3,x_2,x_5,x_4,x_1] - [x_4,x_2,x_5,x_1,x_3] + [x_4,x_2,x_5,x_3,x_1] + [x_4,x_3,x_5,x_1,x_2] - [x_4,x_3,x_5,x_2,x_1],$$

and identity (b) enables us to write $w(3,5)$ in terms of basic commutators:

$$w(3,5) = -[x_2,x_1,x_5;x_4,x_3] + [x_3,x_1,x_5;x_4,x_2] - [x_4,x_1,x_5;x_3,x_2] - [x_3,x_2,x_5;x_4,x_1] + [x_4,x_2,x_5;x_3,x_1] - [x_4,x_3,x_5;x_2,x_1].$$

(A.2.1.2) Relations obtained from $w(3,5)$ and elimination of commutators from these relations.

We use the relations obtained from $W(3,5)$ by the following five substitutions for the variables given in the form of a table: each column of figures gives one substitution. We use of course the notational convention explained in 0.2 (p.4).

$x_1 \to$	1	1	1	1	2
$x_2 \to$	2	2	2	3	3
$x_3 \to$	3	3	4	4	4
$x_4 \to$	4	5	5	5	5
$x_5 \to$	5	4	3	2	1

The relations obtained are:

(r.1): $-[4,3,5;2,1]+[4,2,5;3,1]-[4,1,5;3,2]-[3,2,5;4,1]$
$+[3,1,5;4,2]-[2,1,5;4,3]=0,$

(r,2): $-[5,3,4;2,1]+[5,2,4;3,1]-[5,1,4;3,2]-[3,2,4;5,1]$
$+[3,1,4;5,2]-[2,1,4;5,3]=0,$

(r.3): $-[5,4,3;2,1]+[5,2,3;4,1]-[5,1,3;4,2]-[4,2,3;5,1]$
$+[4,1,3;5,2]-[2,1,3;5,4]=0,$ or
$[4,3,5;2,1]-[5,3,4;2,1]+[5,2,3;4,1]-[5,1,3;4,2]$
$-[4,2,3;5,1]+[4,1,3;5,2]-[2,1,3;5,4]=0$

(r.4): $-[5,4,2;3,1]+[5,3,2;4,1]-[5,1,2;4,3]-[4,3,2;5,1]$
$+[4,1,2;5,3]-[3,1,2;5,4]=0,$ or
$[4,2,5;3,1]-[5,2,4;3,1]-[3,2,5;4,1]+[5,2,3;4,1]$
$-[5,1,2;4,3]+[3,2,4;5,1]-[4,2,3;5,1]+[4,1,2;5,3]$
$-[3,1,2;5,4]=0,$

(r.5): $-[5,4,1;3,2]+[5,3,1;4,2]-[5,2,1;4,3]-[4,3,1;5,2]$
$+[4,2,1;5,3]-[3,2,1;5,4]=0,$ or
$[4,1,5;3,2]-[5,1,4;3,2]-[3,1,5;4,2]+[5,1,3;4,2]$
$+[2,1,5;4,3]-[5,1,2;4,3]+[3,1,4;5,2]-[4,1,3;5,2]$
$-[2,1,4;5,3]+[4,1,2;5,3]+[2,1,3;5,4]-[3,1,2;5,4]=0.$

The alternative forms of (r.3), (r.4), (r.5) are respectively the basic forms of the non-basic expressions of (r.3), (r.4), (r.5). In future similar cases the non-basic forms will not be recorded. Now use relations (r.1), (r.2), (r.3), (r.4) to express the basic commutators $[2,1,5;4,3]$, $[2,1,4;5,3]$, $[2,1,3;5,4]$, $[3,1,2;5,4]$ in terms of other basic commutators. Since each of these basic commutators appear in one and only one of the relations (r.1) - (r.4), we can eliminate them. These basic commutators, however, appear in relation (r.5). A direct substitution will show that relation (r.5) is then identically satisfied so that no other basic commutators can be eliminated.

(A.2.1.3) Computations for the proof of Theorem (2.1.1).

The following table gives the images of the terms of the homogeneous five generator word v in $\overline{W}^5_{5,3}$ under the homomorphisms φ_i, $i \in \{1,2,\ldots,8\}$ (p.21) in terms of basic commutators in three generators. An explanation of the table and its use follows.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7		φ_8
$e_{103}d_{103} \rightarrow$	$e_{103}d_{28}$	$e_{103}d_{29}$	$e_{103}d_{30}$	x	x	x	x		x
$e_{104}d_{104} \rightarrow$	$e_{104}d_{28}$	$e_{104}d_{29}$	$e_{104}d_{36}$	x	x	x	x		x
$e_{105}d_{105} \rightarrow$	$e_{105}d_{28}$	$e_{105}d_{35}$	x	x	x	x	x		x
$e_{106}d_{106} \rightarrow$	$e_{106}d_{34}$	x	x	x	x	x	x		x
$e_{107}d_{107} \rightarrow$	0	$e_{107}d_{42}$	0	0	0	0	$e_{107}d_{39}$		$-e_{107}d_{42}$
$e_{108}d_{108} \rightarrow$	$e_{108}d_{41}$	$e_{108}d_{42}$	0	0	0	0	$e_{108}d_{37}$		0
$e_{109}d_{109} \rightarrow$	0	$e_{109}d_{42}$	$e_{109}d_{48}$	$e_{109}d_{37}$	$e_{109}d_{40}$	0	0		0
$e_{110}d_{110} \rightarrow$	$e_{110}d_{41}$	$e_{110}d_{42}$	$e_{110}d_{48}$	$e_{110}d_{39}$	$e_{110}d_{40}$	0	0		$e_{110}d_{47}$
$e_{111}d_{111} \rightarrow$	0	0	$e_{111}d_{54}$	$e_{111}d_{37}$	$e_{111}d_{40}$	0	$-e_{111}d_{39}$		$e_{111}d_{51}$
$e_{112}d_{112} \rightarrow$	0	0	$e_{112}d_{54}$	$e_{112}d_{39}$	$e_{112}d_{40}$	0	$-e_{112}d_{37}$		$e_{112}d_{53}$
$e_{113}d_{113} \rightarrow$	0	0	$e_{113}d_{48}$	$e_{113}d_{37}$	$e_{113}d_{45}$	0	$-e_{113}d_{44}$		$e_{113}d_{42}$
$e_{114}d_{114} \rightarrow$	$e_{114}d_{41}$	$e_{114}d_{47}$	$e_{114}d_{48}$	$e_{114}d_{39}$	$e_{114}d_{46}$	$e_{114}d_{38}$	0		$e_{114}d_{42}$
$e_{115}d_{115} \rightarrow$	0	$e_{115}d_{51}$	$e_{115}d_{54}$	$e_{115}d_{37}$	$e_{115}d_{45}$	0	0		0
$e_{116}d_{116} \rightarrow$	0	$e_{116}d_{53}$	$e_{116}d_{54}$	$e_{116}d_{39}$	$e_{116}d_{46}$	$e_{116}d_{38}$	$e_{116}d_{49}$		0
$e_{118}d_{118} \rightarrow$	0	$e_{118}d_{53}$	0	0	$e_{118}d_{52}$	x	x		x
$e_{119}d_{119} \rightarrow$	0	0	$e_{119}d_{48}$	$e_{119}d_{44}$	$e_{119}d_{45}$	0	$-e_{119}d_{37}$		$e_{119}d_{47}$
$e_{120}d_{120} \rightarrow$	0	$e_{120}d_{47}$	$e_{120}d_{48}$	$e_{120}d_{44}$	$e_{120}d_{46}$	$e_{120}d_{43}$	$-e_{120}d_{37}$	$+e_{120}d_{39}$	0
$e_{121}d_{121} \rightarrow$	$e_{121}d_{50}$	$e_{121}d_{51}$	$e_{121}d_{54}$	$e_{121}d_{44}$	$e_{121}d_{45}$	0	0		$e_{121}d_{53}$
$e_{122}d_{122} \rightarrow$	$e_{122}d_{50}$	$e_{122}d_{53}$	$e_{122}d_{54}$	$e_{122}d_{44}$	$e_{122}d_{46}$	$e_{122}d_{43}$	0		$e_{122}d_{51}$
$e_{124}d_{124} \rightarrow$	$e_{124}d_{50}$	$e_{124}d_{53}$	0	$e_{124}d_{49}$	x	x	x		x

The commutators that may occur in v, with their coefficients, are listed in the left-most column. Their images in $F_3(\underline{\underline{U}})$ under φ_i are listed in the i-th column, headed φ_i. By way of example:

$$\text{under } \varphi_1, \; d_{103} = [2,1,3,4,5] \to [2,1,2,2,3] = d_{28};$$

$$\text{under } \varphi_7, \; d_{120} = [4,2,3;5,1] \to [3,2,1;2,1] =$$

$$-[2,1,3;2,1] + [3,1,2;2,1] = -d_{37} + d_{39}.$$

Each column adds up to zero in $F_3(\underline{\underline{U}})$, because by assumption the image $v\varphi_i$ of v under each homomorphism φ_i is zero in $F_3(\underline{\underline{U}})$. In general, not all the basic commutators in $v\varphi_i$ are distinct. However, by collecting the like terms together (which is possible since the commutators commute with each other) we can express $v\varphi_i$ as a linear combination with integral coefficients of distinct basic commutators. As the d_i $(25 \leq i \leq 54)$ are independent in $F_3(\underline{\underline{U}})$ the coefficient of each such basic commutator in $F_3(\underline{\underline{U}})$ is zero.

Sometimes there is only one term d_j which under φ_i gives a particular basic commutator in $F_3(\underline{\underline{U}})$. Then the coefficient e_j of this d_j is immediately seen to be zero, and so this term d_j need not be considered any longer in the further maps φ_{i+1} ... . It then suffices to enter no more than an x in the table.

We thus obtain the following set of linear equations satisfied by the integers e_i $(103 \leq i \leq 124)$.

Equations obtained from the column headed by

(a) φ_1: (1) $e_{103} + e_{104} + e_{105} = 0,$

(2) $e_{108} + e_{110} + e_{114} = 0,$

(3) $e_{121} + e_{122} + e_{124} = 0,$

(4) $e_{106} = 0.$

(b) φ_2: (5) $e_{107} + e_{108} + e_{109} + e_{110} = 0,$

(6) $e_{116} + e_{118} + e_{122} + e_{124} = 0,$

(7) $e_{103} + e_{104} = 0,$

(8) $e_{114} + e_{120} = 0,$

(9) $e_{115} + e_{121} = 0,$

(10) $e_{105} = 0.$

(c) φ_3: (11) $e_{109} + e_{110} + e_{113} + e_{114} + e_{119} + e_{120} = 0,$

(12) $e_{111} + e_{112} + e_{115} + e_{116} + e_{121} + e_{122} = 0,$

(13) $e_{103} = 0,$

(14) $e_{104} = 0.$

(d) φ_4: (15) $e_{109} + e_{111} + e_{113} + e_{115} = 0,$

(16) $e_{110} + e_{112} + e_{114} + e_{116} = 0,$

(17) $e_{119} + e_{120} + e_{121} + e_{122} = 0,$

(18) $e_{124} = 0.$

(e) φ_5: (19) $e_{109} + e_{110} + e_{111} + e_{112} = 0,$

(20) $e_{113} + e_{115} + e_{119} + e_{121} = 0,$

(21) $e_{114} + e_{116} + e_{120} + e_{122} = 0,$

(22) $e_{118} = 0.$

(f) φ_6: (23) $e_{114} + e_{116} = 0,$

(24) $e_{120} + e_{122} = 0.$

(g) φ_7: (25) $e_{108} - e_{112} - e_{119} - e_{120} = 0,$

(26) $e_{107} - e_{111} + e_{120} = 0,$

(27) $- e_{113} = 0,$

(28) $e_{116} = 0.$

(h) φ_8: (29) $-e_{107} + e_{113} + e_{114} = 0,$

(30) $e_{110} + e_{119} = 0,$

(31) $e_{111} + e_{122} = 0,$

(32) $e_{112} + e_{121} = 0.$

Equations (4), (10), (13), (14), (18), (22), (27), (28) together with (8), (23), (24), (29) and (31) give at once that all the following coefficients are zero: e_{103}, e_{104}, e_{105}, e_{106}, e_{107}, e_{111}, e_{113}, e_{114}, e_{116}, e_{118}, e_{120}, e_{122}, e_{124}. Using this the equations (2), (3), (9), (12), (15), (16), (17) reduce to the following system with coefficient matrix as shown:

	e_{108}	e_{109}	e_{110}	e_{112}	e_{115}	e_{119}	e_{121}
(2)	1		1				
(3)							1
(9)					1		1
(12)				1	1		1
(15)		1			1		
(16)			1	1			
(17)						1	1

From this it is immediate that the coefficients e_{108}, e_{109}, e_{110}, e_{112}, e_{115}, e_{119}, e_{121} are also zero.

(A.2.2.1) $w_1(2,5) = [x_1,x_2;x_3,x_4;x_5]$ in terms of basic commutators.

$$\begin{aligned} w_1(2,5) &= [x_2,x_1;x_4,x_3;x_5] = -[x_4,x_3;x_5;x_2,x_1] - [x_5;x_2x_1;x_4,x_3] \\ &= -[x_4,x_3,x_5;x_2,x_1] + [x_2,x_1,x_5;x_4,x_3]. \end{aligned}$$

(A.2.2.2) $2w_2(2,5)$ (p.23) is a consequence of $w_1(2,5)$, $w(3,5)$ (p.19).

First we note that $w(3,5)$ given below is a law in $F_3(\underline{\underline{N}}_5)$ and hence in $F_2(\underline{\underline{N}}_5)$.

$$\begin{aligned} w(3,5) = &-[x_4,x_1,x_5;x_3,x_2] + [x_4,x_2,x_5;x_3,x_1] - [x_4,x_3,x_5;x_2,x_1] \\ &+ [x_3,x_1,x_5;x_4,x_2] - [x_3,x_2,x_5;x_4,x_1] - [x_2,x_1,x_5;x_4,x_3]. \end{aligned}$$

Now, from $w_1(2,5)$ we deduce that

$$\begin{aligned} v_1 = -w_1(2,5) &= [x_4,x_3,x_5;x_2,x_1] - [x_2,x_1,x_5;x_4,x_3], \\ v_2 &= -[x_4,x_2,x_5;x_3,x_1] + [x_3,x_1,x_5;x_4,x_2], \\ v_3 &= [x_4,x_1,x_5;x_3,x_2] - [x_3,x_2,x_5;x_4,x_1] \end{aligned}$$

are laws in $F_2(\underline{\underline{N}}_5)$. Hence

$$\begin{aligned} -(v_1+v_2+v_3+w(3,5)) &= 2(-[x_3,x_1,x_5;x_4,x_2] + [x_3,x_2,x_5;x_4,x_1] + [x_2,x_1,x_5;x_4,x_3]) \\ &= 2w_2(2,5) \end{aligned}$$

is a law in $F_2(\underline{\underline{N}}_5)$.

(A.2.2.3) Relations obtained from $w_1(2,5)$, $w_2(2,5)$ (p.23) in $F_5(\underline{\underline{U}}_1)$ and elimination of commutators from them.

Since in $F_5(\underline{\underline{U}}_1)$, $w_1(2,5)$ is a law, the following relations hold in $F_5(\underline{\underline{U}}_1)$.

(1) $[1,2;3,4;5] = 1$, (2) $[1,3;2,4;5] = 1$, (3) $[1,4;2,3;5] = 1$,

(4) $[1,2;3,5;4] = 1$, (5) $[1,3;2,5;4] = 1$, (6) $[1,5;2,3;4] = 1$,

(7) $[1,2;4,5;3] = 1$, (8) $[1,4;2,5;3] = 1$, (9) $[1,5;2,4;3] = 1$,

(10) $[1,3;4,5;2] = 1$, (11) $[1,4;3,5;2] = 1$, (12) $[1,5;3,4;2] = 1$,

(13) $[2,3;4,5;1] = 1$, (14) $[2,4;3,5;1] = 1$, (15) $[2,5;3,4;1] = 1$.

In terms of basic commutators these reduce to:

(1) $[2,1,5;4,3] = [4,3,5;2,1]$

(2) $[4,2,5;3,1] = [3,1,5;4,2]$

(3) $[4,1,5;3,2] = [3,2,5;4,1]$

(4) $[2,1,4;5,3] = [5,3,4;2,1]$

(5) $[3,1,4;5,2] = [5,2,4;3,1]$

(6) $[5,1,4;3,2] = [3,2,4;5,1]$

(7) $[2,1,3;5,4] = - [4,3,5;2,1] + [5,3,4;2,1]$

(8) $[5,2,3;4,1] = [4,1,3;5,2]$

(9) $[5,1,3;4,2] = [4,2,3;5,1]$

(10) $[3,1,2;5,4] = - [4,2,5;3,1] + [5,2,4;3,1]$

(11) $[4,1,2;5,3] = - [3,2,5;4,1] + [5,2,3;4,1]$

(12) $[5,1,2;4,3] = - [3,2,4;5,1] + [4,2,3;5,1]$

(13) $- [2,1,3;5,4] + [3,1,2;5,4] = - [4,1,5;3,2] + [5,1,4;3,2]$

(14) $- [2,1,4;5,3] + [4,1,2;5,3] = - [3,1,5;4,2] + [5,1,3;4,2]$

(15) $- [2,1,5;4,3] + [5,1,2;4,3] = - [3,1,4;5,2] + [4,1,3;5,2]$.

From these relations we eliminate the following basic commutators.

$d_{107} = [4,3,5;2,1]$, $d_{108} = [5,3,4;2,1]$, $d_{109} = [4,2,5;3,1]$,

$d_{110} = [5,2,4;3,1]$, $d_{111} = [4,1,5;3,2]$, $d_{112} = [5,1,4;3,2]$,

$d_{114} = [5,2,3;4,1]$, $d_{116} = [5,1,3;4,2]$, $d_{118} = [5,1,2;4,3]$,

$d_{120} = [4,2,3;5,1]$, $d_{124} = [4,1,2;5,3]$, $d_{125} = [2,1,3;5,4]$,

$d_{126} = [3,1,2;5,4]$.

Similarly, $w_2(2,5)$ is a law in $F_5(\underline{\underline{U}}_1)$ and we derive (by permutation of the generators) immediately the following relations in $F_5(\underline{\underline{U}}_1)$.

(1) $[3,2,5;4,1] - [3,1,5;4,2] + [2,1,5;4,3] = 0$,

(2) $[3,2,4;5,1] - [3,1,4;5,2] + [2,1,4;5,3] = 0$,

From these two relations we eliminate the basic commutators:

$d_{117} = [2,1,5;4,3]$, $d_{123} = [2,1,4;5,3]$.

(A.2.2.4.) Table giving the images of the terms of v under the homomorphisms φ_{1i}, $i \in \{1,2,3\}$ (p.25) in terms of basic commutators in 2 generators.

	φ_{11}	φ_{12}	φ_{13}
$e_4[2,1,1,3] \to$	$e_4 \ d_{23}$	$e_4 \ d_{20}$	x
$e_5[2,1,2,3] \to$	$e_5 \ d_{24}$	0	$-e_5 \ d_{20}$
$e_6[2,1,3,3] \to$	0	$e_6 \ d_{21}$	x
$e_7[3,1,1,2] \to$	$e_7 \ d_{20}$	x	x
$e_8[3,1,2,2] \to$	$e_8 \ d_{21}$	x	x
$e_9[3,1,2,3] \to$	0	0	$-e_9 \ d_{21}$
$e_{10}[3,1;2,1] \to$	$e_{10} \ d_{23}$	$-e_{10} \ d_{23}$	x
$e_{11}[3,2;2,1] \to$	$e_{11} \ d_{24}$	0	$e_{11} \ d_{23}$
$e_{12}[3,2;3,1] \to$	0	$-e_{12} \ d_{24}$	x

Equations obtained from the column headed by

(a) φ_{11}: (1) $e_4 + e_{10} = 0$,

(2) $e_5 + e_{11} = 0$,

(3) $e_7 = 0$,

(4) $e_8 = 0$.

(b) φ_{12}: (5) $e_4 = 0$,

(6) $e_6 = 0$,

(7) $e_{10} = 0$,

(8) $e_{12} = 0$.

(c) φ_{13}: (9) $e_5 = 0$,

(10) $e_9 = 0$,

(11) $e_{11} = 0$.

From equations (1) - (11), we see immediately that $e_i = 0$ for $i \in \{4,5,\ldots,12\}$.

(A.2.2.5.) <u>Table giving the images of the terms of</u> v <u>under the homomorphisms</u> φ_{2i}, $i \in \{1,2,\ldots,6\}$ (<u>p.26</u>), <u>in terms of basic commutators in</u> 2 <u>generators</u>.

		φ_{21}	φ_{22}	φ_{23}	φ_{24}	φ_{25}	φ_{26}
$e_{13}[2,1,3,4]$	$\to$	$e_{13}\,d_{19}$	x	x	x	x	x
$e_{14}[3,1,2,4]$	$\to$	0	$e_{14}\,d_{19}$	x	x	x	x
$e_{15}[4,1,2,3]$	$\to$	0	0	$e_{15}\,d_{19}$	x	x	x
$e_{16}[4,3;2,1]$	$\to$	0	0	0	$-e_{16}\,d_{24}$	$e_{16}\,d_{23}$	$e_{16}\,d_{24}$
$e_{17}[4,2;3,1]$	$\to$	0	0	0	0	0	$e_{17}\,d_{24}$
$e_{18}[4,1;3,2]$	$\to$	0	0	0	$-e_{18}\,d_{24}$	$-e_{18}\,d_{23}$	0

Relations obtained from the column headed by

(a) φ_{21}: (1) $e_{13} = 0$,

(b) φ_{22}: (2) $e_{14} = 0$,

(c) φ_{23}: (3) $e_{15} = 0$,

(d) φ_{24}: (4) $e_{16} + e_{18} = 0$,

(e) φ_{25}: (5) $e_{16} - e_{18} = 0$,

(f) φ_{26}: (6) $e_{16} + e_{17} = 0$.

Equations (4), (5) give that $e_{16} = 0$, $e_{18} = 0$ and this in turn together with equation (6) give that $e_{16} = 0$. Hence $e_i = 0$ for $i \in \{13,14,\ldots,18\}$.

(A.2.2.6.) <u>Table giving the images of the terms of</u> v <u>under the homomorphisms</u> φ_{3i}, $i \in \{1,2,\ldots,9\}$ <u>(p. 27)</u> <u>in terms of basic commutators in</u> 2 <u>generators.</u>

	φ_{31}	φ_{32}	φ_{33}	φ_{34}	φ_{35}	φ_{36}	φ_{37}	φ_{38}	φ_{39}
$e_{103}[2,1,3,4,5] \to$	$e_{103}d_{19}$	x	x	x	x	x	x	x	x
$e_{104}[3,1,2,4,5] \to$	0	$e_{104}d_{19}$	x	x	x	x	x	x	x
$e_{105}[4,1,2,3,5] \to$	0	0	$e_{105}d_{19}$	x	x	x	x	x	x
$e_{106}[5,1,2,3,4] \to$	0	0	0	$e_{106}d_{19}$	x	x	x	x	x
$e_{113}[3,2,5;4,1] \to$	0	0	0	0	0	0	0	$-e_{113}d_{23}$	x
$e_{115}[3,1,5;4,2] \to$	0	0	0	0	0	0	0	0	$e_{115}d_{23}$
$e_{119}[3,2,4;5,1] \to$	0	0	0	0	0	0	$e_{119}d_{23}$	x	x
$e_{121}[3,1,4;5,2] \to$	0	0	0	0	0	$e_{121}d_{24}$	x	x	x
$e_{122}[4,1,3;5,2] \to$	0	0	0	0	$e_{122}d_{23}$	x	x	x	x

Equations obtained from the column headed by

(a) φ_{31}: (1) $e_{103} = 0$,

(b) φ_{32}: (2) $e_{104} = 0$,

(c) φ_{33}: (3) $e_{105} = 0$,

(d) φ_{34}: (4) $e_{106} = 0$,

(e) φ_{35}: (5) $e_{122} = 0$,

(f) φ_{36}: (6) $e_{121} = 0$,

(g) φ_{37} (7) $e_{119} = 0$,

(h) φ_{38}: (8) $e_{113} = 0$,

(i) φ_{39}: (9) $e_{115} = 0$.

From equations (a) - (i), we note immediately that the coefficients given in the left-most column of the above table are all zero.

Appendix 3: Computations for Chapter 3.

(A.3.1.1) The law $w(4,6)$ (p.32) in terms of basic commutators:

$$
\begin{aligned}
w(4,6) = {} & 2[x_6,x_5;x_2,x_1;x_4,x_3] - 2[x_6,x_5;x_3,x_1;x_4,x_2] + 2[x_6,x_5;x_3,x_2;x_4,x_1] - \\
& [x_4,x_3;x_2,x_1;x_6,x_5] + [x_4,x_2;x_3,x_1;x_6,x_5] - [x_4,x_1;x_3,x_2;x_6,x_5] - \\
& 2[x_6,x_4;x_3,x_2;x_5,x_1] + 2[x_6,x_3;x_4,x_2;x_5,x_1] - 2[x_6,x_2;x_4,x_3;x_5,x_1] + \\
& 2[x_6,x_4;x_3,x_1;x_5,x_2] - 2[x_6,x_3;x_4,x_1;x_5,x_2] + 2[x_6,x_1;x_4,x_3;x_5,x_2] - \\
& 2[x_6,x_4;x_2,x_1;x_5,x_3] + 2[x_6,x_2;x_4,x_1;x_5,x_3] - 2[x_6,x_1;x_4,x_2;x_5,x_3] + \\
& 2[x_6,x_3;x_2,x_1;x_5,x_4] - 2[x_6,x_2;x_3,x_1;x_5,x_4] + 2[x_6,x_1;x_3,x_2;x_5,x_4] - \\
& [x_5,x_4;x_3,x_2;x_6,x_1] + [x_5,x_3;x_4,x_2;x_6,x_1] - [x_5,x_2;x_4,x_3;x_6,x_1] + \\
& [x_5,x_4;x_3,x_1;x_6,x_2] - [x_5,x_3;x_4,x_1;x_6,x_2] + [x_5,x_1;x_4,x_3;x_6,x_2] - \\
& [x_5,x_4;x_2,x_1;x_6,x_3] + [x_5,x_2;x_4,x_1;x_6,x_3] - [x_5,x_1;x_4,x_2;x_6,x_3] + \\
& [x_5,x_3;x_2,x_1;x_6,x_4] - [x_5,x_2;x_3,x_1;x_6,x_4] + [x_5,x_1;x_3,x_2;x_6,x_4].
\end{aligned}
$$

(A.3.1.2) The laws $w_1(4,6)$, $w_2(4,6)$

$$
\begin{aligned}
w_1(4,6) = {} & -2[x_6,x_5;x_2,x_1;x_4,x_3] + 2[x_6,x_5;x_3,x_1;x_4,x_2] - 2[x_6,x_5;x_3,x_2;x_4,x_1] + \\
& [x_4,x_3;x_2,x_1;x_6,x_5] - [x_4,x_2;x_3,x_1;x_6,x_5] + [x_4,x_1;x_3,x_2;x_6,x_5] - \\
& [x_5,x_4;x_3,x_2;x_6,x_1] + [x_5,x_3;x_4,x_2;x_6,x_1] - [x_5,x_2;x_4,x_3;x_6,x_1] + \\
& [x_5,x_4;x_3,x_1;x_6,x_2] - [x_5,x_3;x_4,x_1;x_6,x_2] + [x_5,x_1;x_4,x_3;x_6,x_2] - \\
& [x_5,x_4;x_2,x_1;x_6,x_3] + [x_5,x_2;x_4,x_1;x_6,x_3] - [x_5,x_1;x_4,x_2;x_6,x_3] + \\
& [x_5,x_3;x_2,x_1;x_6,x_4] - [x_5,x_2;x_3,x_1;x_6,x_4] + [x_5,x_1;x_3,x_2;x_6,x_4];
\end{aligned}
$$

$$
\begin{aligned}
w_2(4,6) = {} & -[x_6,x_4;x_2,x_1;x_5,x_3] + [x_6,x_4;x_3,x_1;x_5,x_2] - [x_6,x_4;x_3,x_2;x_5,x_1] + \\
& [x_6,x_5;x_2,x_1;x_4,x_3] - [x_6,x_5;x_3,x_1;x_4,x_2] + [x_6,x_5;x_3,x_2;x_4,x_1] + \\
& [x_5,x_4;x_3,x_2;x_6,x_1] - [x_5,x_4;x_3,x_1;x_6,x_2] + [x_5,x_4;x_2,x_1;x_6,x_3].
\end{aligned}
$$

(A.3.1.3) Relations obtained from $w_1(4,6)$ and $w_2(4,6)$ and elimination of basic commutators from them.

We obtain the following relations from $w_2(4,6)$ by permutation of variables.

(1) $-[6,4;2,1;5,3] + [6,4;3,1;5,2] - [6,4;3,2;5,1] + [6,5;2,1;4,3]$
$-[6,5;3,1;4,2] + [6,5;3,2;4,1] + [5,4;3,2;6,1] - [5,4;3,1;6,2]$
$+[5,4;2,1;6,3] = 0,$

(2) $-[6,3;2,1;5,4] + [6,3;4,1;5,2] - [6,3;4,2;5,1] - [6,5;2,1;4,3]$
$+[4,1;3,2;6,5] - [6,5;3,2;4,1] - [4,2;3,1;6,5] + [6,5;3,1;4,2]$
$+[5,3;4,2;6,1] - [5,3;4,1;6,2] + [5,3;2,1;6,4] = 0,$

(3) $-[6,2;4,1;5,3] + [6,2;3,1;5,4] + [6,2;4,3;5,1] + [4,1;3,2;6,5]$
$-[6,5;3,2;4,1] + [6,5;3,1;4,2] + [4,3;2,1;6,5] - [6,5;2,1;4,3]$
$-[5,2;4,3;6,1] - [5,2;3,1;6,4] + [5,2;4,1;6,3] = 0,$

(4) $[6,1;4,2;5,3] - [6,1;4,3;5,2] - [6,1;3,2;5,4]$
$-[4,2;3,1;6,5] + [6,5;3,1;4,2] + [4,3;2,1;6,5] - [6,5;2,1;4,3]$
$-[6,5;3,2;4,1] + [5,1;3,2;6,4] + [5,1;4,3;6,2] - [5,1;4,2;6,3] = 0,$

and we obtain from $w_1(4,6)$ the relation:

(5) $-2[6,5;2,1;4,3] + 2[6,5;3,1;4,2] - 2[6,5;3,2;4,1] + [4,3;2,1;6,5]$
$-[4,2;3,1;6,5] + [4,1;3,2;6,5] - [5,4;3,2;6,1] + [5,3;4,2;6,1]$
$-[5,2;4,3;6,1] + [5,4;3,1;6,2] - [5,3;4,1;6,2] + [5,1;4,3;6,2]$
$-[5,4;2,1;6,3] + [5,2;4,1;6,3] - [5,1;4,2;6,3] + [5,3;2,1;6,4]$
$-[5,2;3,1;6,4] + [5,1;3,2;6,4].$

From these relations we eliminate the following commutators, one from each relation:

$$d_{578} = [6,4;3,2;5,1],\quad d_{579} = [6,3;4,2;5,1],\quad d_{580} = [6,2;4,3;5,1],$$
$$d_{583} = [6,1;4,3;5,2],\quad d_{604} = [4,1;3,2;6,5].$$

(A.3.1.4) <u>To show that commutators in</u> v_4 <u>given below have trivial coefficients</u>.

$$\begin{aligned} v_4 = {} & e_{575}[6,5;3,2;4,1] + e_{576}[6,5;3,1;4,2] + e_{577}[6,5;2,1;4,3] + e_{590}[5,4;3,2;6,1] \\ & + e_{591}[5,3;4,2;6,1] + e_{592}[5,2;4,3;6,1] + e_{593}[5,4;3,1;6,2] + e_{594}[5,3;4,1;6,2] \\ & + e_{595}[5,1;4,3;6,2] + e_{596}[5,4;2,1;6,3] + e_{597}[5,2;4,1;6,3] + e_{598}[5,1;4,2;6,3] \\ & + e_{599}[5,3;2,1;6,4] + e_{600}[5,2;3,1;6,4] + e_{601}[5,1;3,2;6,4] + e_{602}[4,3;2,1;6,5] \\ & + e_{603}[4,2;3,1;6,5]. \end{aligned}$$

We map v_4 by the following endomorphisms of $F_6(\underline{\underline{U}})$. Each time we use Lemma (3.1.5) to deduce that some of the commutators in v_4 have trivial coefficients. Such commutators of course will not be considered in the next mapping.

		φ'_1	φ'_2	φ'_3	φ'_4	φ'_5	φ'_6	φ'_7
1	→	1	1	1	1	1	1	1
2	→	2	2	2	2	2	2	2
3	→	3	3	3	3	6	3	3
4	→	6	4	6	4	3	4	5
5	→	4	6	5	6	4	6	4
6	→	5	5	4	5	5	5	6

We have:

(a) $$v_4\varphi_1 = e_{575}[5,4;3,2;6,1] + e_{576}[5,4;3,1;6,2] + e_{577}[5,4;2,1;6,3]$$
$$-e_{590}[6,4;3,2;5,1] - e_{591}[6,2;4,3;5,1] - e_{592}[6,3;4,2;5,1]$$
$$-e_{593}[6,4;3,1;5,2] - e_{594}[6,1;4,3;5,2] - e_{595}[6,3;4,1;5,2]$$
$$-e_{596}[6,4;2,1;5,3] - e_{597}[6,1;4,2;5,3] - e_{598}[6,2;4,1;5,3]$$
$$-e_{599}[4,3;2,1;6,5] - e_{600}[4,2;3,1;6,5] - e_{601}[4,1;3,2;6,5]$$
$$+e_{602}[6,3;2,1;5,4] + e_{603}[6,2;3,1;5,4],$$

so that by Lemma (3.1.5), $e_{594} = e_{597} = 0$;

(b) $$v_4\varphi_2 = -e_{575}[6,5;3,2;4,1] - e_{576}[6,5;3,1;4,2] - e_{577}[6,5;2,1;4,3]$$
$$+e_{590}[6,4;3,2;5,1] + e_{591}[6,3;4,2;5,1] + e_{592}[6,2;4,3;5,1]$$
$$+e_{593}[6,4;3,1;5,2] + e_{595}[6,1;4,3;5,2] + e_{596}[6,4;2,1;5,3]$$
$$+e_{598}[6,1;4,2;5,3] + e_{599}[6,3;2,1;5,4] + e_{600}[6,2;3,1;5,4]$$
$$+e_{601}[6,1;3,2;5,4] - e_{602}[4,3;2,1;6,5] - e_{603}[4,2;3,1;6,5],$$

and so $e_{591} = e_{592} = e_{599} = e_{600} = 0$.

(c) $$v_4\varphi_3 = -e_{575}[5,4;3,2;6,1] - e_{576}[5,4;3,1;6,2] - e_{577}[5,4;2,1;6,3]$$
$$-e_{590}[6,5;3,2;4,1] - e_{593}[6,5;3,1;4,2] - e_{595}[6,3;4,2;5,1]$$
$$+e_{595}[5,1;4,2;6,3] - e_{596}[6,5;2,1;4,3] - e_{598}[6,2;4,3;5,1]$$
$$+e_{598}[5,1;4,3;6,2] - e_{601}[5,1;3,2;6,4] - e_{602}[6,3;2,1;5,4]$$
$$e_{603}[6,2;3,1;5,4],$$

and so $e_{595} = e_{598} = e_{602} = e_{603} = 0$;

(d) $$v_4\varphi_4 = -e_{575}[6,5;3,2;4,1] - e_{576}[6,5;3,1;4,2] - e_{577}[6,5;2,1;4,3]$$
$$+e_{590}[6,4;3,2;5,1] + e_{593}[6,4;3,1;5,2] + e_{596}[6,4;2,1;5,3]$$
$$e_{601}[6,1;3,2;5,4],$$

and so $e_{601} = 0$;

(e) $$v_4\varphi_5 = -e_{575}[6,2;3,1;5,4] + e_{575}[5,4;3,1;6,2] - e_{576}[6,1;3,2;5,4]$$
$$+e_{576}[5,4;3,2;6,1] - e_{577}[5,4;2,1;6,3] - e_{590}[6,2;4,3;5,1]$$
$$-e_{593}[6,1;4,3;5,2] - e_{596}[4,3;2,1;6,5],$$

and so $e_{575} = e_{576} = e_{590} = e_{593} = 0$;

(f) $$v_4\varphi_6 = -e_{577}[6,5;2,1;4,3] + e_{596}[6,4;2,1;5,3],$$

and so $e_{596} = 0$;

(g) $$v_4\varphi_7 = e_{577}[6,4;2,1;5,3],$$

and so $e_{577} = 0$.

(A.3.2.1) The law u in terms of basic commutators.

$$\begin{aligned}
u = &-[x_4,x_1,x_5;x_3.x_2;x_6] + [x_4,x_2,x_5;x_3,x_1;x_6] - [x_4,x_3,x_5;x_2,x_1;x_6] \\
&+[x_3,x_1,x_5;x_4,x_2;x_6] - [x_3,x_2,x_5;x_4,x_1;x_6] - [x_2,x_1,x_5;x_4,x_3;x_6] \\
= &\ [x_3,x_2;x_6;x_4,x_1,x_5] - [x_4,x_1,x_5;x_6;x_3,x_2] - [x_3,x_1;x_6;x_4,x_2,x_5] \\
&+[x_4,x_2,x_5;x_6;x_3,x_1] + [x_2,x_1;x_6;x_4,x_3,x_5] - [x_4,x_3,x_5;x_6;x_2,x_1] \\
&-[x_4,x_2;x_6;x_3,x_1,x_5] + [x_3,x_1,x_5;x_6;x_4,x_2] + [x_4,x_1;x_6;x_3,x_2,x_5] \\
&-[x_3,x_2,x_5;x_6;x_4,x_1] + [x_4,x_3;x_6;x_2,x_1,x_5] - [x_2,x_1,x_5;x_6;x_4,x_3]
\end{aligned}$$

$$= -[x_4,x_1,x_5;x_3,x_2,x_6] + [x_4,x_2,x_5;x_3,x_1,x_6] - [x_4,x_3,x_5;x_2,x_1,x_6]$$
$$-[x_4,x_2,x_6;x_3,x_1,x_5] + [x_4,x_1,x_6;x_3,x_2,x_5] + [x_4,x_3,x_6;x_2,x_1,x_5]$$
$$-[x_4,x_1,x_5,x_6;x_3,x_2] + [x_4,x_2,x_5,x_6;x_3,x_1] - [x_4,x_3,x_5,x_6;x_2,x_1]$$
$$+[x_3,x_1,x_5,x_6;x_4,x_2] - [x_3,x_2,x_5,x_6;x_4,x_1] - [x_2,x_1,x_5,x_6;x_4,x_3] .$$

(A.3.2.2) Relations obtained from $w_1(3,6)$ and elimination of basic commutators from them.

(1) [5,4,6;2,1,3] + [4,3,5;2,1,6] - [5,3,4;2,1,6] - [5,2,6;4,1,3]
+[5,2,3;4,1,6] + [5,1,6;4,2,3] - [5,1,3;4,2,6] = 0,

(2) -[5,4,6;2,1,3] + [6,4,5;2,1,3] + [5,3,6;2,1,4] - [6,3,5;2,1,4]
-[6,2,4;5,1,3] + [6,2,3;5,1,4] + [6,1,4;5,2,3] - [6,1,3;5,2,4] = 0

(3) -[6,4,5;2,1,3] - [4,3,6;2,1,5] + [6,3,4;2,1,5] + [6,2,5;4,1,3]
-[6,2,3;4,1,5] - [6,1,5;4,2,3] + [6,1,3;4,2,5] = 0,

(4) -[5,4,6;2,1,3] + [5,3,6;2,1,4] - [4,3,6;2,1,5] + [5,4,6;3,1,2]
+[5,2,6;3,1,4] + [4,2,6;3,1,5] + [4,1,5;3,2,6] - [5,1,4;3,2,6]
-[5,3,6;4,1,2] + [5,2,6;4,1,3] + [5,1,3;4,2,6] - [5,1,2;4,3,6] = 0,

(5) [5,4,6;2,1,3] - [6,4,5;2,1,3] + [6,3,4;2,1,5] - [5,3,4;2,1,6]
-[5,4,6;3,1,2] + [6,4,5;3,1,2] - [6,2,4;3,1,5] + [5,2,4;3,1,6]
+[5,1,6;3,2,4] - [6,1,5;3,2,4] - [6,3,4;5,1,2] + [6,2,4;5,1,3]
+[6,1,3;5,2,4] - [6,1,2;5,3,4] = 0,

(6) [6,4,5;2,1,3] - [6,3,5;2,1,4] + [4,3,5;2,1,6] - [6,4,5;3,1,2]
+[6,2,5;3,1,4] - [4,2,5;3,1,6] - [4,1,6;3,2,5] + [6,1,4;3,2,5]
+[6,3,5;4,1,2] - [6,2,5;4,1,3] - [6,1,3;4,2,5] + [6,1,2;4,3,5] = 0,

(7) $-[5,4,6;3,1,2] - [4,2,5;3,1,6] + [5,2,4;3,1,6] + [5,1,6;3,2,4]$
$-[4,1,6;3,2,5] + [5,3,6;4,1,2] - [5,2,3;4,1,6] - [5,1,6;4,2,3]$
$+[5,1,2;4,3,6] = 0,$

(8) $[5,4,6;3,1,2] - [6,4,5;3,1,2] - [5,2,6;3,1,4] + [6,2,5;3,1,4]$
$+[6,1,4;3,2,5] - [5,1,4;3,2,6] + [6,3,4;5,1,2] - [6,2,3;5,1,4]$
$-[6,1,4;5,2,3] + [6,1,2;5,3,4] = 0,$

(9) $[4,2,6;3,1,5] - [6,2,4;3,1,5] + [6,4,5;3,1,2] - [6,1,5;3,2,4]$
$+[4,1,5;3,2,6] - [6,3,5;4,1,2] + [6,2,3;4,1,5] + [6,1,5;4,2,3]$
$-[6,1,2;4,3,5] = 0,$

(10) $[5,3,6;2,1,4] - [5,3,4;2,1,6] - [5,2,6;3,1,4] + [5,2,4;3,1,6]$
$+[5,1,6;3,2,4] - [5,1,4;3,2,6] = 0,$

(11) $-[6,3,5;2,1,4] + [6,3,4;2,1,5] + [6,2,5;3,1,4] - [6,2,4;3,1,5]$
$+[6,1,4;3,2,5] - [6,1,5;3,2,4] = 0,$

(12) $[5,4,6;2,1,3] - [5,3,6;2,1,4] + [4,3,6;2,1,5] - [5,4,6;3,1,2]$
$+[5,2,6;3,1,4] - [4,2,6;3,1,5] - [4,1,5;3,2,6] + [5,1,4;3,2,6]$
$+[5,3,6;4,1,2] - [5,2,6;4,1,3] - [5,1,3;4,2,6] + [5,1,2;4,3,6] = 0,$

(13) $[5,3,6;2,1,4] - [6,3,5;2,1,4] - [4,3,6;2,1,5] + [6,3,4;2,1,5]$
$+[4,3,5;2,1,6] - [5,3,4;2,1,6] - [5,3,6;4,1,2] + [6,3,5;4,1,2]$
$-[6,2,3;4,1,5] + [5,2,3;4,1,6] + [5,1,6;4,2,3] - [6,1,5;4,2,3]$
$+[6,1,2;4,3,5] - [5,1,2;4,3,6] - [6,3,4;5,1,2] + [6,2,3;5,1,4]$
$+[6,1,4;5,2,3] - [6,1,2;5,3,4] = 0,$

(14) $[5,3,6;4,1,2] - [6,3,5;4,1,2] - [5,2,6;4,1,3] + [6,2,5;4,1,3]$
$+[6,1,3;4,2,5] - [5,1,3;4,2,6] - [6,1,2;4,3,5] + [5,1,2;4,3,6]$
$+[6,3,4;5,1,2] - [6,2,4;5,1,3] - [6,1,3;5,2,4] + [6,1,2;5,3,4] = 0,$

(15) $-[6,4,5;3,1,2] - [4,2,6;3,1,5] + [6,2,4;3,1,5] + [6,1,5;3,2,4]$
$-[4,1,5;3,2,6] + [6,3,5;4,1,2] - [6,2,3;4,1,5] - [6,1,5;4,2,3]$
$+[6,1,2;4,3,5] = 0,$

(16) $[4,3,6;2,1,5] - [4,3,5;2,1,6] - [4,2,6;3,1,5] + [4,2,5;3,1,6]$
$+[4,1,6;3,2,5] - [4,1,5;3,2,6] = 0.$

From these relations we eliminate:

$d_{509} = [6,1,4;3,2,5]$, $d_{510} = [4,1,5;3,2,6]$, $d_{511} = [5,1,4;3,2,6]$,
$d_{520} = [6,1,3;4,2,5]$, $d_{521} = [5,1,3;4,2,6]$, $d_{522} = [6,1,2;4,3,5]$,
$d_{523} = [5,1,2;4,3,6]$, $d_{528} = [6,1,3;5,2,4]$, $d_{529} = [6,1,2;5,3,4]$.

(A.3.2.3) Relations obtained from $w_2(3,6)$ and elimination of basic commutators from them.

By Lemma (3.0.1), (iii), sums of commutators of type (2,2,2) with arbitrary entries form a subgroup of $F_6(\underline{\underline{N}}_6)$, hence also of $F_6(\underline{\underline{U}})$. We denote this subgroup of $F_6(\underline{\underline{U}})$ by Γ and the following computations are modulo Γ, that is, we throw away commutators of type (2,2,2) whenever they occur.

(1) $-[5,1,2,6;4,3] + [6,1,2,5;4,3] + [4,1,2,6;5,3] - [6,1,2,4;5,3]$
$-[3,1,2,6;5,4] + [6,1,2,3;5,4] - [4,1,2,5;6,3] + [5,1,2,4;6,3]$
$+[3,1,2,5;6,4] - [5,1,2,3;6,4] - [3,1,2,4;6,5] + [4,1,2,3;6,5] = 0 \bmod \Gamma,$

(2) $-[5,1,3,6;4,2] + [6,1,3,5;4,2] + [4,1,3,6;5,2] - [6,1,3,4;5,2]$
$-[2,1,3,6;5,4] + [6,1,2,3;5,4] - [4,1,3,5;6,2] + [5,1,3,4;6,2]$
$+[2,1,3,5;6,4] - [5,1,2,3;6,4] - [2,1,3,4;6,5] + [4,1,2,3;6,5] = 0 \bmod \Gamma,$

(3) $-[5,2,3,6;4,1] + [6,2,3,5;4,1] + [4,2,3,6;5,1] - [6,2,3,4;5,1]$
$+[6,1,2,3;5,4] - [4,2,3,5;6,1] + [5,2,3,4;6,1] - [5,1,2,3;6,4]$
$+[4,1,2,3;6,5] = 0 \bmod \Gamma,$

(4) $-[5,1,4,6;3,2] + [6,1,4,5;3,2] + [3,1,4,6;5,2] - [6,1,3,4;5,2]$
$-[2,1,4,6;5,3] + [6,1,2,4;5,3] - [3,1,4,5;6,2] + [5,1,3,4;6,2]$
$+[2,1,4,5;6,3] - [5,1,2,4;6,3] - [2,1,3,4;6,5] + [3,1,2,4;6,5] = 0 \bmod \Gamma,$

(5) $-[5,2,4,6;3,1] + [6,2,4,5;3,1] + [3,2,4,6;5,1] - [6,2,3,4;5,1]$
$+[6,1,2,4;5,3] - [3,2,4,5;6,1] + [5,2,3,4;6,1] - [5,1,2,4;6,3]$
$+[3,1,2,4;6,5] = 0 \bmod \Gamma,$

(6) $-[5,3,4,6;2,1] + [6,3,4,5;2,1] - [6,2,3,4;5,1] + [6,1,3,4;5,2]$
$+[5,2,3,4;6,1] - [5,1,3,4;6,2] + [2,1,3,4;6,5] = 0 \bmod \Gamma,$

(7) $-[4,1,5,6;3,2] + [6,1,4,5;3,2] + [3,1,5,6;4,2] - [6,1,3,5;4,2]$
$-[2,1,5,6;4,3] + [6,1,2,5;4,3] - [3,1,4,5;6,2] + [4,1,3,5;6,2]$
$+[2,1,4,5;6,3] - [4,1,2,5;6,3] - [2,1,3,5;6,4] + [3,1,2,5;6,4] = 0 \bmod \Gamma,$

(8) $-[4,2,5,6;3,1] + [6,2,4,5;3,1] + [3,2,5,6;4,1] - [6,2,3,5;4,1]$
$+[6,1,2,5;4,3] - [3,2,4,5;6,1] + [4,2,3,5;6,1] - [4,1,2,5;6,3]$
$+[3,1,2,5;6,4] = 0 \bmod \Gamma,$

(9) $-[4,3,5,6;2,1] + [6,3,4,5;2,1] - [6,2,3,5;4,1] + [6,1,3,5;4,2]$
$+[4,2,3,5;6,1] - [4,1,3,5;6,2] + [2,1,3,5;6,4] + [2,1,3,5;6,4] = 0 \bmod \Gamma,$

(10) $[6,3,4,5;2,1] - [6,2,4,5;3,1] + [6,1,4,5;3,2] + [3,2,4,5;6,1]$
$-[3,1,4,5;6,2] + [2,1,4,5;6,3] = 0 \bmod \Gamma,$

(11) $-[4,1,5,6;3,2] + [5,1,4,6;3,2] + [3,1,5,6;4,2] - [5,1,3,6;4,2]$
$-[2,1,5,6;4,3] + [5,1,2,6;4,3] - [3,1,4,6;5,2] + [4,1,3,6;5,2]$
$+[2,1,4,6;5,3] - [4,1,2,6;5,3] - [2,1,3,6;5,4] + [3,1,2,6;5,4] = 0 \bmod \Gamma,$

(12) $-[4,2,5,6;3,1] + [5,2,4,6;3,1] + [3,2,5,6;4,1] - [5,2,3,6;4,1]$
$+[5,1,2,6;4,3] - [3,2,4,6;5,1] + [4,2,3,6;5,1] - [4,1,2,6;5,3]$
$+[3,1,2,6;5,4] = 0 \bmod \Gamma$,

(13) $-[4,3,5,6;2,1] + [5,3,4,6;2,1] - [5,2,3,6;4,1] + [5,1,3,6;4,2]$
$+[4,2,3,6;5,1] - [4,1,3,6;5,2] + [2,1,3,6;5,4] = 0 \bmod \Gamma$,

(14) $[5,3,4,6;2,1] - [5,2,4,6;3,1] + [5,1,4,6;3,2] + [3,2,4,6;5,1]$
$-[3,1,4,6;5,2] + [2,1,4,6;5,3] = 0 \bmod \Gamma$,

(15) $[4,3,5,6;2,1] - [4,2,5,6;3,1] + [4,1,5,6;3,2] + [3,2,5,6;4,1]$
$-[3,1,5,6;4,2] + [2,1,5,6;4,3] = 0 \bmod \Gamma$,

From these relations we eliminate:

$d_{545} = [2,1,5,6;4,3]$, $d_{554} = [2,1,4,6;5,3]$, $d_{557} = [2,1,3,6;5,4]$,
$d_{558} = [3,1,2,6;5,4]$, $d_{566} = [2,1,4,5;6,3]$, $d_{569} = [2,1,3,5;6,4]$,
$d_{570} = [3,1,2,5;6,4]$, $d_{572} = [2,1,3,4;6,5]$, $d_{573} = [3,1,2,4;6,5]$,
$d_{574} = [4,1,2,3;6,5]$.

(A.3.2.4) Relations obtained from $w_3(3,6)$, $w_4(3,6)$, $w_5(3,6)$ and elimination of basic commutators from them.

The following relations are obtained from $w_3(3,6)$ by permutation of variables.

(1) $[6,5;2,1;4,3] - [6,4;2,1;5,3] + [6,3;2,1;5,4] + [5,4;2,1;6,3]$
$-[5,3;2,1;6,4] + [4,3;2,1;6,5] = 0$,

(2) $[6,5;3,1;4,2] - [6,4;3,1;5,2] + [6,2;3,1;5,4] + [5,4;3,1;6,2]$

$-[5,2;3,1;6,4] + [4,2;3,1;6,5] = 0,$

(3) $[6,5;3,2;4,1] - [6,4;3,2;5,1] + [6,1;3,2;5,4] + [5,4;3,2;6,1]$

$-[5,1;3,2;6,4] + [4,1;3,2;6,5] = 0,$

(4) $[6,5;3,2;4,1] - [6,3;4,1;5,2] + [6,2;4,1;5,3] + [5,3;4,1;6,2]$

$-[5,2;4,1;6,3] - 2[4,1;3,2;6,5] = 0,$

(5) $[6,5;3,1;4,2] - [6,3;4,2;5,1] + [6,1;4,2;5,3] + [5,3;4,2;6,1]$

$-[5,1;4,2;6,3] - 2[4,2;3,1;6,5] = 0,$

(6) $[6,5;2,1;4,3] - [6,2;4,3;5,1] + [6,1;4,3;5,2] + [5,2;4,3;6,1]$

$-[5,1;4,3;6,2] - 2[4,3;2,1;6,5] = 0,$

(7) $[6,4;3,2;5,1] - [6,3;4,2;5,1] + [6,2;4,3;5,1] - 2[5,1;4,3;6,2]$

$+2[5,1;4,2;6,3] - 2[5,1;3,2;6,4] = 0,$

(8) $[6,4;3,1;5,2] - [6,3;4,1;5,2] + [6,1;4,3;5,2] - 2[5,2;4,3;6,1]$

$+2[5,2;4,1;6,3] - 2[5,2;3,1;6,4] = 0,$

(9) $[6,4;2,1;5,3] - [6,2;4,1;5,3] + [6,1;4,2;5,3] - 2[5,3;4,2;6,1]$

$+2[5,3;4,1;6,2] - 2[5,3;2,1;6,4] = 0,$

(10) $[6,3;2,1;5,4] - [6,2;3,1;5,4] + [6,1;3,2;5,4] - 2[5,4;3,2;6,1]$

$+2[5,4;3,1;6,2] - 2[5,4;2,1;6,3] = 0,$

(11) $[5,4;3,2;6,1] - [5,3;4,2;6,1] + [5,2;4,3;6,1] - 2[6,1;4,3;5,2]$

$+2[6,1;4,2;5,3] - 2[6,1;3,2;5,4] = 0,$

(12) $[5,4;3,1;6,2] - [5,3;4,1;6,2] + [5,1;4,3;6,2] - 2[6,2;4,3;5,1]$

$+2[6,2;4,1;5,3] - 2[6,2;3,1;5,4] = 0,$

(13) $[5,4;2,1;6,3] - [5,2;4,1;6,3] + [5,1;4,2;6,3] - 2[6,3;4,2;5,1]$

$+2[6,3;4,1;5,2] - 2[6,3;2,1;5,4] = 0,$

(14) $[5,3;2,1;6,4] - [5,2;3,1;6,4] + [5,1;3,2;6,4] - 2[6,4;3,2;5,1]$
$+2[6,4;3,1;5,2] - 2[6,4;2,1;5,3] = 0,$

(15) $[4,3;2,1;6,5] - [4,2;3,1;6,5] + [4,1;3,2;6,5] - 2[6,5;3,2;4,1]$
$+2[6,5;3,1;4,2] - 2[6,5;2,1;4,3] = 0,$

The following relations are obtained by repeated application of the operation of addition to the relations obtained from $w_3(3,6)$, $w_4(3,6)$, $w_5(3,6)$.

(16) $[5,4;3,1;6,2] - [5,3;4,1;6,2] - [5,1;4,3;6,2] - [5,4;2,1;6,3]$
$+[5,2;4,1;6,3] + [5,1;4,2;6,3] + [5,3;2,1;6,4] - [5,2;3,1;6,4]$
$-[5,1;3,2;6,4] - [4,3;2,1;6,5] + [4,2;3,1;6,5] + [4,1;3,2;6,5] = 0,$

(17) $-[5,4;3,2;6,1] + [5,3;4,2;6,1] - [5,2;4,3;6,1] + 2[5,1;4,3;6,2]$
$-2[5,1;4,2;6,3] + 2[5,1;3,2;6,4] + [4,3;2,1;6,5] - [4,2;3,1;6,5]$
$-[4,1;3,2;6,5] = 0,$

(18) $-2[5,4;3,2;6,1] + 2[5,3;4,1;6,2] + 2[5,1;4,3;6,2] - 2[5,2;4,1;6,3]$
$-2[5,1;4,2;6,3] - [5,3;2,1;6,4] + [5,2;3,1;6,4] + 3[5,1;3,2;6,4]$
$+[4,3;2,1;6,5] - [4,2;3,1;6,5] - 3[4,1;3,2;6,5] = 0,$

(19) $-2[5,3;4,2;6,1] + 2[5,3;4,1;6,2] + [5,4;2,1;6,3] - [5,2;4,1;6,3]$
$+[5,1;4,2;6,3] - 2[5,3;2,1;6,4] + [4,3;2,1;6,5] + [4,2;3,1;6,5]$
$-[4,1;3,2;6,5] = 0.$

From relations (1) - (15) we eliminate:

$d_{575} = [6,5;3,2;4,1]$, $d_{576} = [6,5;3,1;4,2]$, $d_{577} = [6,5;2,1;4,3]$,
$d_{578} = [6,4;3,2;5,1]$, $d_{579} = [6,3;4,2;5,1]$, $d_{580} = [6,2;4,3;5,1]$,
$d_{581} = [6,4;3,1;5,2]$, $d_{584} = [6,4;2,1;5,3]$, $d_{587} = [6,3;2,1;5,4]$,
$d_{592} = [5,2;4,3;6,1]$,

and from relations (16) - (19) we eliminate:

$$d_{593} = [5,4;3,1;6,2], \quad d_{596} = [5,4;2,1;6,3],$$

$$d_{599} = [5,3;2,1;6,4]. \quad d_{604} = [4,1;3,2;6,5]$$

(A.3.2.5) Table giving the images of the left-normed terms of u_1 (p.44) under φ_{1i} $i \in \{1,2,3\}$ (p.45) in terms of basic commutators in 3 generators.

		φ_{11}	φ_{12}	φ_{13}
e_{55}[2,1,1,3,4]	→	0	[2,1,1,2,3]	[2,1,1,3,3]
e_{56}[2,1,2,3,4]	→	0	[2,1,2,2,3]	[2,1,2,3,3]
e_{57}[2,1,3,3,4]	→	0	[2,1,2,2,3]	[2,1,3,3,3]
e_{58}[2,1,3,4,4]	→	0	[2,1,2,3,3]	x
e_{59}[3,1,1,2,4]	→	[2,1,1,1,3]	[2,1,1,2,3]	[3,1,1,2,3]
e_{60}[3,1,2,2,4]	→	[2,1,1,1,3]	[2,1,2,2,3]	[3,1,2,2,3]
e_{61}[3,1,2,3,4]	→	[2,1,1,2,3]	x	x
e_{62}[3,1,2,4,4]	→	[2,1,1,3,3]	x	x
e_{63}[4,1,1,2,3]	→	[3,1,1,1,2]	[3,1,1,2,2]	x
e_{64}[4,1,2,2,3]	→	[3,1,1,1,2]	[3,1,2,2,2]	x
e_{65}[4,1,2,3,3]	→	[3,1,1,2,2]	x	x
e_{66}[4,1,2,3,4]	→	[3,1,1,2,3]	x	x

We deduce immediately from the column headed by φ_{11} $e_{61} = e_{62} = e_{65} = e_{66} = 0$, from the column headed by φ_{12} $e_{58} = e_{63} = e_{64}$ and from the column headed by φ_{13} $e_{55} = e_{56} = e_{57} = e_{59} = e_{60} = 0$. Hence $e_i = 0$ for all $i \in \{55,56,\ldots,66\}$.

(A.3.2.6) <u>Table giving the images of the terms of type</u> (3,2) <u>of</u> u_1 (p.44) <u>under</u> φ_{2i}, $i \in \{1,2,3,4\}$ p.45) <u>in terms of basic commutators in</u> 3 <u>generators</u>.

		φ_{21}	φ_{22}	φ_{23}	φ_{24}
$e_{67}[3,1,4;2,1]$	$\to$	$e_{67}d_{217}$	x	x	x
$e_{68}[3,2,4;2,1]$	$\to$	$e_{68}d_{218}$	x	x	x
$e_{69}[4,1,3;2,1]$	$\to$	$e_{69}d_{181}$	x	x	x
$e_{70}[4,2,3;2,1]$	$\to$	$e_{70}d_{182}$	x	x	x
$e_{71}[4,3,3;2,1]$	$\to$	$e_{71}d_{183}$	x	x	x
$e_{72}[4,3,4;2,1]$	$\to$	0	0	$-e_{72}d_{183}$	x
$e_{73}[2,1,4;3,1]$	$\to$	0	$-e_{73}d_{219}$	x	x
$e_{74}[3,2,4;3,1]$	$\to$	$e_{74}d_{220}$	x	x	x
$e_{75}[4,1,2;3,1]$	$\to$	$e_{75}d_{194}$	x	x	x
$e_{76}[4,2,2;3,1]$	$\to$	$e_{76}d_{196}$	x	x	x
$e_{77}[4,2,3;3,1]$	$\to$	$e_{77}d_{197}$	x	x	x
$e_{78}[4,2,4;3,1]$	$\to$	0	0	$e_{78}d_{161}$	x
$e_{79}[2,1,4;3,2]$	$\to$	0	$-e_{79}d_{222}$	x	x
$e_{80}[3,1,4;3,2]$	$\to$	$e_{80}d_{222}$	x	x	x
$e_{81}[4,1,1;3,2]$	$\to$	$e_{81}d_{205}$	x	x	x

		φ_{21}	φ_{22}	φ_{23}	φ_{24}
$e_{82}[4,1,2;3,2]$	$\rightarrow$	$e_{82}d_{206}$	x	x	x
$e_{83}[4,1,3;3,2]$	$\rightarrow$	$e_{83}d_{207}$	x	x	x
$e_{84}[4,1,4;3,2]$	$\rightarrow$	0	0	$e_{84}d_{165}$	x
$e_{85}[2,1,3;4,1]$	$\rightarrow$	$e_{85}d_{156}$	$-e_{85}d_{168}$	x	x
$e_{86}[3,1,2;4,1]$	$\rightarrow$	$e_{86}d_{158}$	x	x	x
$e_{87}[3,2,2;4,1]$	$\rightarrow$	$e_{87}d_{160}$	x	x	x
$e_{88}[3,2,3;4,1]$	$\rightarrow$	$e_{88}d_{161}$	x	x	x
$e_{89}[3,2,4;4,1]$	$\rightarrow$	0	0	$e_{89}d_{197}$	x
$e_{90}[4,2,3;4,1]$	$\rightarrow$	0	0	$e_{90}d_{220}$	x
$e_{91}[2,1,3;4,2]$	$\rightarrow$	$e_{91}d_{162}$	$-e_{91}d_{169}$	x	x
$e_{92}[3,1,1;4,2]$	$\rightarrow$	$e_{92}d_{163}$	x	x	x
$e_{93}[3,1,2;4,2]$	$\rightarrow$	$e_{93}d_{164}$	x	x	x
$e_{94}[3,1,3;4,2]$	$\rightarrow$	$e_{94}d_{165}$	x	x	x
$e_{95}[3,1,4;4,2]$	$\rightarrow$	0	0	$e_{95}d_{207}$	x
$e_{96}[4,1,3;4,2]$	$\rightarrow$	0	0	$e_{96}d_{222}$	x
$e_{97}[2,1,1;4,3]$	$\rightarrow$	$-e_{97}d_{156}$	$-e_{97}d_{159}$	x	x
$e_{98}[2,1,2;4,3]$	$\rightarrow$	$-e_{98}d_{162}$	$-e_{98}d_{165}$	x	x
$e_{99}[2,1,3;4,3]$	$\rightarrow$	0	$-e_{99}d_{170}$	x	x
$e_{100}[2,1,4;4,3]$	$\rightarrow$	0	0	0	$e_{100}d_{207}$
$e_{101}[3,1,2;4,3]$	$\rightarrow$	$e_{101}d_{169}$	x	x	x
$e_{102}[4,1,2;4,3]$	$\rightarrow$	0	0	$-e_{102}d_{109}$	x

Equations obtained from the column headed by φ_{21}, φ_{22}, φ_{23}, φ_{24} give immediately that $e_i = 0$ for all $i \in \{67,68,\ldots,102\}$.

(A.3.2.7) Table giving the images of the terms of w (p.49) under φ_{3i}, $i = 1,2,3$ in terms of basic commutators in 3 generators.

		φ_{31}	φ_{32}	φ_{33}
$e_{289}[4,3,4;2,1,4]$	$\to$	0	$e_{289}[323;213]$	0
$e_{317}[4,2,4;3,1,4]$	$\to$	$e_{317}[313;213]$	$e_{317}[323;213]$	$[323;313]$
$e_{329}[4,1,4;3,2,4]$	$\to$	$e_{329}[313;213]$	0	$-[323;313]$
$e_{334}[4,3,4;4,1,2]$	$\to$	$e_{334}[323;311]$	x	x
$e_{337}[4,2,4;4,1,3]$	$\to$	$e_{337}[313;312]$	$e_{337}[323;312]$	x
$e_{338}[4,2,3;4,1,4]$	$\to$	$-e_{338}[313;312]$	$e_{338}[322;313]$	x

Relations obtained from the column headed by

(a) φ_{31} : (1) $e_{317} + e_{329} = 0$

(2) $e_{337} - e_{338} = 0$

(3) $e_{334} = 0$

(b) φ_{32} : (4) $e_{289} + e_{317} = 0$

(5) $e_{337} = 0$

(6) $e_{338} = 0$

(c) φ_{33} (7) $e_{317} - e_{329} = 0$.

Equations (1) and (7) give at once $e_{317} = e_{329} = 0$; equation (4) then gives $e_{289} = 0$. Hence all coefficients are trivial.

(A.3.2.8) Table giving the images of the terms of v_2 (p.49) under φ_{4i}, $i \in \{1,2,\dots,10\}$, in terms of basic commutators in 4 generators.

		φ_{41}	φ_{42}	φ_{43}	φ_{44}	φ_{45}
$e_{490}d_{490}$	$\to$	0	0	0	0	$e_{490}d_{255}$
$e_{491}d_{491}$	$\to$	0	0	0	0	$e_{491}d_{257}$
$e_{492}d_{492}$	$\to$	0	0	0	0	$e_{492}d_{255}$
$e_{493}d_{493}$	$\to$	0	0	0	0	$e_{493}d_{257}$
$e_{494}d_{494}$	$\to$	0	0	0	0	0
$e_{495}d_{495}$	$\to$	0	0	0	0	$e_{495}d_{270}$
$e_{496}d_{496}$	$\to$	0	0	0	0	0
$e_{497}d_{497}$	$\to$	0	0	0	0	$e_{497}d_{279}$
$e_{498}d_{498}$	$\to$	0	$e_{498}d_{255}$	$-e_{498}d_{255}$	$-e_{498}d_{255}+e_{498}d_{257}$	x
$e_{499}d_{499}$	$\to$	0	$e_{499}d_{257}$	$-e_{499}d_{255}+e_{499}d_{257}$	x	x
$e_{500}d_{500}$	$\to$	0	$e_{500}d_{255}$	0	$e_{500}d_{270}$	0
$e_{501}d_{501}$	$\to$	0	$e_{501}d_{257}$	$e_{501}d_{270}$	0	0
$e_{502}d_{502}$	$\to$	0	0	$e_{502}d_{255}$	$e_{502}d_{279}$	0
$e_{503}d_{503}$	$\to$	0	$e_{503}d_{270}$	x	x	x
$e_{504}d_{504}$	$\to$	0	0	$e_{504}d_{279}$	$e_{504}d_{255}$	x
$e_{505}d_{505}$	$\to$	0	$e_{505}d_{279}$	x	x	x
$e_{506}d_{506}$	$\to$	0	$e_{506}d_{255}$	0	$e_{506}d_{270}$	$-e_{506}d_{255}$
$e_{507}d_{507}$	$\to$	0	$e_{507}d_{257}$	$e_{507}d_{270}$	0	$-e_{507}d_{257}$
$e_{508}d_{508}$	$\to$	0	0	$e_{508}d_{255}$	$e_{508}d_{279}$	0
$e_{512}d_{512}$	$\to$	$e_{512}d_{255}$	0	$e_{512}d_{279}$	$e_{512}d_{279}+e_{512}d_{291}$	x
$e_{513}d_{513}$	$\to$	$e_{513}d_{257}$	0	$e_{513}d_{279}+e_{513}d_{291}$	x	x

		φ_{41}	φ_{42}	φ_{43}	φ_{44}	φ_{45}
$e_{514}d_{514}$	$\to$	$e_{514}d_{255}$	0	0	$e_{514}d_{297}$	0
$e_{515}d_{515}$	$\to$	$e_{515}d_{257}$	0	$e_{515}d_{297}$	0	0
$e_{516}d_{516}$	$\to$	$e_{516}d_{270}$	x	x	x	x
$e_{517}d_{517}$	$\to$	$e_{517}d_{279}$	x	x	x	x
$e_{518}d_{518}$	$\to$	$e_{518}d_{255}$	0	0	$e_{518}d_{297}$	$-e_{518}d_{255}$
$e_{519}d_{519}$	$\to$	$e_{519}d_{257}$	0	$e_{519}d_{297}$	0	$-e_{519}d_{257}$
$e_{524}d_{524}$	$\to$	$e_{524}d_{291}$	$e_{524}d_{279}+e_{524}d_{291}$	x	x	x
$e_{525}d_{525}$	$\to$	$e_{525}d_{291}$	$e_{525}d_{297}$	0	0	$e_{525}d_{279}+e_{525}d_{291}$
$e_{526}d_{526}$	$\to$	$e_{526}d_{297}$	x	x	x	x
$e_{527}d_{527}$	$\to$	$e_{527}d_{291}$	$e_{527}d_{297}$	0	0	$-e_{527}d_{270}+e_{527}d_{297}$

	φ_{46}	φ_{47}	φ_{48}	φ_{49}	φ_{410}
$e_{490}d_{490}$	0	$-e_{490}d_{255}$	$-e_{490}d_{262}+e_{490}d_{264}$	x	x
$e_{491}d_{491}$	$e_{491}d_{270}$	$-e_{491}d_{255}+e_{491}d_{257}$	x	x	x
$e_{492}d_{492}$	$-e_{492}d_{255}$	0	$e_{492}d_{274}$	$e_{492}d_{277}$	x
$e_{493}d_{493}$	$-e_{493}d_{255}+e_{493}d_{257}$	x	x	x	x
$e_{494}d_{494}$	$-e_{494}d_{255}$	$e_{494}d_{255}$	$e_{494}d_{283}$	0	$e_{494}d_{289}$
$e_{495}d_{495}$	x	x	x	x	x
$e_{496}d_{496}$	$-e_{496}d_{279}$	$e_{496}d_{279}$	$e_{496}d_{262}$	x	x
$e_{497}d_{497}$	x	x	x	x	x
$e_{498}d_{498}$	x	x	x	x	x
$e_{499}d_{499}$	x	x	x	x	x
$e_{500}d_{500}$	$e_{500}d_{279}$	0	$e_{500}d_{274}$	$e_{500}d_{309}$	x
$e_{501}d_{501}$	$e_{501}d_{279}+e_{501}d_{291}$	x	x	x	x
$e_{502}d_{502}$	$e_{502}d_{279}$	0	$e_{502}d_{283}$	$e_{502}d_{312}$	x
$e_{503}d_{503}$	x	x	x	x	x
$e_{504}d_{504}$	x	x	x	x	x
$e_{505}d_{505}$	x	x	x	x	x
$e_{506}d_{506}$	0	0	0	$e_{506}d_{324}$	x
$e_{507}d_{507}$	$-e_{507}d_{270}+e_{507}d_{297}$	x	x	x	x
$e_{508}d_{508}$	0	$-e_{508}d_{255}$	0	$-e_{508}d_{306}$	x
$e_{512}d_{512}$	x	x	x	x	x
$e_{513}d_{513}$	x	x	x	x	x
$e_{514}d_{514}$	0	$e_{514}d_{279}$	$e_{514}d_{301}$	x	x
$e_{515}d_{515}$	0	$e_{515}d_{279}+e_{515}d_{291}$	x	x	x
$e_{516}d_{516}$	x	x	x	x	x

	φ_{46}	φ_{47}	φ_{48}	φ_{49}	φ_{410}
$e_{517}d_{517}$	x	x	x	x	x
$e_{518}d_{518}$	0	0	$e_{518}d_{319}$	x	x
$e_{519}d_{519}$	$-e_{519}d_{270}$	$-e_{519}d_{270}+e_{519}d_{297}$	x	x	x
$e_{524}d_{524}$	x	x	x	x	x
$e_{525}d_{525}$	x	x	x	x	x
$e_{526}d_{526}$	x	x	x	x	x
$e_{527}d_{527}$	x	x	x	x	x

Equations (we only produce those which give immediately that some coefficients are trivial) obtained from the column headed by:

(a) φ_{41} : $e_{516} = 0$, $e_{517} = 0$, $e_{526} = 0$;

(b) φ_{42} : $e_{503} = 0$, $e_{505} = 0$, $e_{524} = 0$;

(c) φ_{43} : $e_{499} = 0$, $e_{513} = 0$;

(d) φ_{44} : $e_{498} = 0$, $e_{504} = 0$, $e_{512} = 0$;

(e) φ_{45} : $e_{495} = 0$, $e_{525} = 0$, $e_{527} = 0$;

(f) φ_{46} : $e_{493} = 0$, $e_{497} = 0$, $e_{501} = 0$, $e_{507} = 0$;

(g) φ_{47} : $e_{491} = 0$, $e_{515} = 0$, $e_{519} = 0$;

(h) φ_{48} : $e_{490} = 0$, $e_{496} = 0$, $e_{514} = 0$, $e_{518} = 0$;

(i) φ_{49} : $e_{492} = 0$, $e_{500} = 0$, $e_{502} = 0$, $e_{506} = 0$, $e_{508} = 0$;

(j) φ_{410} : $e_{494} = 0$.

Hence the commutators in v_2 have trivial coefficients.

(A.3.2.9) <u>An expression for</u> v_4 (p.49) <u>in basic commutators not including the commutators that have been eliminated.</u>

$$
\begin{aligned}
v_4 = {} & e_{582}[6,3;4,1;5,2] + e_{583}[6,1;4,3;5,2] + e_{585}[6,2;4,1;5,3] \\
& + e_{586}[6,1;4,2;5,3] + e_{588}[6,2;3,1;5,4] + e_{589}[6,1;3,2;5,4] \\
& + e_{590}[5,4;3,2;6,1] + e_{591}[5,3;4,2;6,1] + e_{594}[5,3;4,1;6,2] \\
& + e_{595}[5,1;4,3;6,2] + e_{597}[5,2;4,1;6,3] + e_{598}[5,1;4,2;6,3] \\
& + e_{600}[5,2;3,1;6,4] + e_{601}[5,1;3,2;6,4] + e_{602}[4,3;2,1;6,5] \\
& + e_{603}[4,2;3,1;6,5].
\end{aligned}
$$

(A.3.2.10) <u>Table giving the images of the terms of</u> v_4 (p.49) <u>under</u> φ_{6i}, $i \in \{1,2,\ldots,11\}$ (p.54) <u>in terms of basic commutators in</u> 3 <u>generators.</u>

		φ_{61}	φ_{62}	φ_{63}	φ_{64}	φ_{65}
$e_{583}d_{583}$	$\to$	$e_{583}d_{219}$	$e_{583}d_{217}$	0	$-e_{583}d_{221}$	$-e_{583}d_{224}$
$e_{586}d_{586}$	$\to$	$e_{586}d_{219}$	$e_{586}d_{217}$	0	0	$-e_{586}d_{224}$
$e_{589}d_{589}$	$\to$	0	0	0	0	0
$e_{590}d_{590}$	$\to$	0	0	0	0	0
$e_{591}d_{591}$	$\to$	$e_{591}d_{219}$	0	0	$e_{591}d_{221}$	0
$e_{594}d_{594}$	$\to$	$e_{594}d_{219}$	0	$e_{594}d_{223}$	$e_{594}d_{221}$	$e_{594}d_{224}$
$e_{595}d_{595}$	$\to$	$e_{595}d_{219}$	0	0	$-e_{595}d_{221}$	$-e_{595}d_{224}$
$e_{597}d_{597}$	$\to$	$e_{597}d_{219}$	0	$e_{597}d_{223}$	0	$e_{597}d_{224}$
$e_{598}d_{598}$	$\to$	$e_{598}d_{219}$	0	0		$-e_{598}d_{224}$
$e_{600}d_{600}$	$\to$	0	0	$e_{600}d_{223}$	0	0

	φ_{61}	φ_{62}	φ_{63}	φ_{64}	φ_{65}
$e_{601}d_{601}$	0	0	0	0	0
$e_{602}d_{602}$	0	0	0	0	0
$e_{603}d_{603}$	0	0	0	0	0

	φ_{66}	φ_{67}	φ_{68}	φ_{69}	φ_{610}	φ_{611}
$e_{583}d_{583}$	0	0	0	0	0	$-e_{583}d_{218}$
$e_{586}d_{586}$	0	0	0	0	0	0
$e_{589}d_{589}$	0	0	0	0	$-e_{589}d_{218}$	$e_{589}d_{218}$
$e_{590}d_{590}$	0	$e_{590}d_{218}$	0	$-e_{590}d_{218}$	$-e_{590}d_{218}$	0
$e_{591}d_{591}$	0	$e_{591}d_{218}$	0	0	$e_{591}d_{218}$	0
$e_{594}d_{594}$	0	$e_{594}d_{218}$	$-e_{594}d_{218}$	0	0	0
$e_{595}d_{595}$	$e_{595}d_{218}$	0	$-e_{595}d_{218}$	0	0	0
$e_{597}d_{597}$	0	0	0	$e_{597}d_{218}$	0	0
$e_{598}d_{598}$	$e_{598}d_{218}$	0	0	0	0	0
$e_{600}d_{600}$	0	0	0	0	0	0
$e_{601}d_{601}$	0	0	$e_{601}d_{218}$	0	$-e_{601}d_{218}$	0
$e_{602}d_{602}$	0	0	0	0	0	0
$e_{603}d_{603}$	0	0	0	0	0	0

Equations obtained from the column headed by

(a) φ_{61} : (1) $e_{583} + d_{586} + e_{591} + e_{594} + e_{595} + e_{597} + e_{598} = 0,$

(b) φ_{62} : (2) $e_{583} + e_{586} = 0,$

(c) φ_{63} : (3) $e_{594} + e_{597} + e_{600} = 0,$

(d) φ_{64} : (4) $- e_{583} + e_{591} + e_{594} - e_{595} = 0,$

(e) φ_{65} : (5) $- e_{583} - e_{586} + e_{594} - e_{595} + e_{597} - e_{598} = 0,$

(f) φ_{66} : (6) $e_{595} + e_{598} = 0,$

(g) φ_{67} : (7) $e_{590} + e_{591} + e_{594} = 0,$

(h) φ_{68} : (8) $- e_{594} - e_{595} + e_{601} = 0,$

(i) φ_{69} : (9) $- e_{590} + e_{597} = 0,$

(j) φ_{610} : (10) $- e_{589} - e_{590} + e_{591} - e_{601} = 0,$

(k) φ_{611} : (11) $- e_{583} + e_{589} = 0.$

Equations (2), (6), (5) together with equations (1), (3) imply that $e_{600} = 0$, $e_{591} = 0$. Equations (10) and (11) now give:

(12) $$+ e_{583} + e_{590} + e_{601} = 0,$$

and equations (7), (8) give:

(13) $$e_{590} - e_{595} + e_{601} = 0,$$

from which we obtain by subtracting (13) from (12):

(14) $$e_{583} + e_{595} = 0.$$

Hence $e_{594} = 0$, as equations (4), (14) will show, and equations (7), (9) now give $e_{590} = 0$, $e_{597} = 0$. Furthermore, we obtain from equations (2), (10), (11), (6), (8) the following:

(15) $e_{583} = e_{589}$, $e_{586} = -e_{589}$,

(16) $e_{595} = e_{601}$, $e_{598} = -e_{601}$,

(17) $e_{589} = -e_{601}$.

Appendix 4: Computations for Chapter 4.

(A.4.1.1) The law μ (p. 56) of $F_2(\underline{\underline{N}}_6)$ in terms of basic commutators.

$$\mu = [(-[x_3,x_4;x_5;x_1,x_2] + [x_1,x_2;x_5;x_3,x_4]),x_6]$$
$$=-[[x_3,x_4,x_5;x_1,x_2],x_6] + [[x_1,x_2,x_5;x_3,x_4],x_6]$$
$$= [x_1,x_2;x_6;x_3,x_4,x_5] - [x_3,x_4,x_5;x_6;x_1,x_2] - [x_3,x_4;x_6;x_1,x_2,x_5]$$
$$+[x_1,x_2,x_5;x_6;x_3,x_4]$$
$$= [x_2,x_1,x_6;x_4,x_3,x_5] - [x_4,x_3,x_5,x_6;x_2,x_1] - [x_4,x_3,x_6;x_2,x_1,x_5]$$
$$+[x_2,x_1,x_5,x_6;x_4,x_3]$$
$$=-[x_4,x_3,x_5;x_2,x_1,x_6] - [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_3,x_5,x_6;x_2,x_1]$$
$$+[x_2,x_1,x_5,x_6;x_4,x_3].$$

(A.4.1.2) The law ν (p.56) of $F_2(\underline{\underline{N}}_6)$ in terms of basic commutators.

$$\nu = [[x_2,x_1,x_5;x_4,x_3],x_6] + [[x_3,x_2,x_5;x_4,x_1],x_6] - [[x_3,x_1,x_5;x_4,x_2],x_6]$$
$$=-[x_4,x_3;x_6;x_2,x_1,x_5] + [x_2,x_1,x_5;x_6;x_4,x_3] - [x_4,x_1;x_6;x_3,x_2,x_5]$$
$$+[x_3,x_2,x_5;x_6;x_4,x_1] + [x_4,x_2;x_6;x_3,x_1,x_5] - [x_3,x_1,x_5;x_6;x_4,x_2]$$
$$=-[x_4,x_3,x_6;x_2,x_1,x_5] + [x_4,x_2,x_6;x_3,x_1,x_5] - [x_4,x_1,x_6;x_3,x_2,x_5]$$
$$+[x_3,x_2,x_5,x_6;x_4,x_1] - [x_3,x_1,x_5,x_6;x_4,x_2] + [x_2,x_1,x_5,x_6;x_4,x_3]$$

(A.4.1.3) Table giving the commutators of weight 6 in two generators and their coefficients in the expansion of $c_i(T_j)$ (p. 62) with respect to the coefficients $\alpha_3(1)$ and $\alpha_3(2)$.

In the table the coefficients of commutators of weight 6 which are trivial are omitted γ is a constant arising from the expansion with respect to $\alpha_3(1)$, δ is a constant arising from the expansion with respect to $\alpha_3(2)$; the result is obtained by observing that [1,2,2;2,1,1] = -[2,1,2;2,1.1] etc.

Term	relevant coefficient	commutator of weight 6	coefficient
$c_1(T_1)$	$\alpha_3(1)$	[2,1,1;2,1,1]	
$c_1(T_2)$	"	"	
$c_1(T_3)$	"	"	
$c_2(T_1)$	"	[1,2,1;2,1,1]	
$c_2(T_2)$	"	"	
$c_2(T_3)$	"	"	
$c_3(T_1)$	"	[1,2,2;2,1,1]	$\alpha_2(1)\alpha_1(2)\alpha_5(2)\alpha_4(2)\gamma$
$c_3(T_2)$	"	"	$-\alpha_4(1)\alpha_1(2)\alpha_5(2)\alpha_2(2)\gamma$
$c_3(T_3)$	"	"	$\alpha_4(1)\alpha_2(2)\alpha_5(2)\alpha_1(2)\gamma$
$c_4(T_1)$	"	[2,1,2;2,1,1]	$\alpha_2(2)\alpha_1(1)\alpha_5(2)\alpha_4(2)\gamma$
$c_4(T_2)$	"	"	$-\alpha_4(2)\alpha_1(1)\alpha_5(2)\alpha_2(2)\gamma$
$c_4(T_3)$	"	"	$\alpha_4(2)\alpha_2(1)\alpha_5(2)\alpha_1(2)\gamma$
$c_5(T_1)$	$\alpha_3(2)$	[2,1,1;1,2,2]	$\alpha_2(2)\alpha_1(1)\alpha_5(1)\alpha_4(1)\delta$

Table (A.4.1.3) continued.

Term	relevant coefficient	commutator of weight 6	coefficient
$c_5(T_2)$	$\alpha_3(2)$	[2.1.1;1,2,2]	$-\alpha_4(2)\alpha_1(1)\alpha_5(1)\alpha_2(1)\delta$
$c_5(T_3)$	"	"	$\alpha_4(2)\alpha_2(1)\alpha_5(1)\alpha_1(1)\delta$
$c_6(T_1)$	"	[1,2,1;1,2,2]	$\alpha_2(1)\alpha_1(2)\alpha_5(1)\alpha_4(1)\delta$
$c_6(T_2)$	"	"	$-\alpha_4(1)\alpha_1(2)\alpha_5(1)\alpha_2(1)\delta$
$c_6(T_3)$	"	"	$\alpha_4(1)\alpha_2(2)\alpha_5(1)\alpha_1(1)\delta$
$c_7(T_1)$	"	[1,2,2;1,2,2]	
$c_7(T_2)$	"	"	
$c_7(T_3)$	"	"	
$c_8(T_1)$	$\alpha_3(2)$	[2,1,2;1,2,2]	
$c_8(T_2)$	"	"	
$c_8(T_3)$	"	"	
$c_1(T_4)$	$\alpha_3(2)$	[2,1,2,1;2,1]	$\alpha_2(1)\alpha_5(1)\alpha_4(2)\alpha_1(1)\delta$
$c_1(T_5)$	"	"	$-\alpha_1(1)\alpha_5(1)\alpha_4(2)\alpha_2(1)\delta$
$c_1(T_6)$	"	"	$-\alpha_4(1)\alpha_5(1)\alpha_2(2)\alpha_1(1)\delta$
$c_2(T_4)$	$\alpha_3(1)$	[1,2,1,1;2,1]	$\alpha_2(2)\alpha_5(1)\alpha_4(2)\alpha_1(1)\gamma$
$c_2(T_5)$	"	"	$-\alpha_1(2)\alpha_5(1)\alpha_4(2)\alpha_2(1)\gamma$
$c_2(T_6)$	"	"	$-\alpha_4(2)\alpha_5(1)\alpha_2(2)\alpha_1(1)\gamma$
$c_3(T_4)$	"	[1,2,1,2;2,1]	$\alpha_2(2)\alpha_5(2)\alpha_4(2)\alpha_1(1)\gamma$
$c_3(T_5)$	"	"	$-\alpha_1(2)\alpha_5(2)\alpha_4(2)\alpha_2(1)\gamma$
$c_3(T_6)$	"	"	$-\alpha_4(2)\alpha_5(2)\alpha_2(2)\alpha_1(1)\gamma$

Table (A.4.1.3) continued.

Term	relevant coefficient	commutator of weight 6	coefficient
$c_4(T_4)$	$\alpha_3(2)$	[2,1,2,2;2,1]	$\alpha_2(1)\alpha_5(2)\alpha_4(2)\alpha_1(1)\delta$
$c_4(T_5)$	"	"	$-\alpha_1(1)\alpha_5(2)\alpha_4(2)\alpha_2(1)\delta$
$c_4(T_6)$	"	"	$-\alpha_4(1)\alpha_5(2)\alpha_2(2)\alpha_1(1)\delta$
$c_5(T_4)$	"	[2,1,2,1;1,2]	$\alpha_2(1)\alpha_5(1)\alpha_4(1)\alpha_1(2)\delta$
$c_5(T_5)$	"	"	$-\alpha_1(1)\alpha_5(1)\alpha_4(1)\alpha_2(2)\delta$
$c_5(T_6)$	"	"	$-\alpha_4(1)\alpha_5(1)\alpha_2(1)\alpha_1(2)\delta$
$c_6(T_4)$	$\alpha_3(1)$	[1,2,1,1;1,2]	$\alpha_2(2)\alpha_5(1)\alpha_4(1)\alpha_1(2)\gamma$
$c_6(T_5)$	"	"	$-\alpha_1(2)\alpha_5(1)\alpha_4(1)\alpha_2(2)\gamma$
$c_6T_6)$	"	"	$-\alpha_4(2)\alpha_5(1)\alpha_2(1)\alpha_1(2)\gamma$
$c_7(T_4)$	"	[1,2,1,2;1,2]	$\alpha_2(2)\alpha_5(2)\alpha_4(1)\alpha_1(2)\gamma$
$c_7(T_5)$	"	"	$-\alpha_1(2)\alpha_5(2)\alpha_4(1)\alpha_2(2)\gamma$
$c_7(T_6)$	"	"	$-\alpha_4(2)\alpha_5(2)\alpha_2(1)\alpha_1(2)\gamma$
$c_8(T_4)$	$\alpha_3(2)$	[2,1,2,2;1,2]	$\alpha_2(1)\alpha_5(2)\alpha_4(1)\alpha_1(2)\delta$
$c_8(T_5)$	"	"	$-\alpha_1(1)\alpha_5(2)\alpha_4(1)\alpha_2(2)\delta$
$c_8(T_6)$	"	"	$-\alpha_4(1)\alpha_5(2)\alpha_2(1)\alpha_1(2)\delta$

(A.4.1.4) <u>Relations obtained from</u> $w_1(2,6)$ <u>involving basic commutators in precisely the generators</u> 1,2,3,4.

The following two relations are obtained from $w_7(2,6)$ by the substitutions:

$$
\begin{array}{lll}
x_1 \to 1 & & 1 \\
x_2 \to 2 & & 2 \\
x_3 \to 3 & & 3 \\
x_4 \to 4 & & 4 \\
x_5 \to 1 & & 2
\end{array}
$$

(1) $[3,1,4;2,1] - [4,1,3;2,1] - [2,1,4;3,1] + [4,1,2;3,1] - [4,1,1;3,2] + [2,1,3;4,1] - [3,1,2;4,1] + [3,1,1;4,2] - [2,1,1;4,3] = 0$ mod $(F_6(\underline{\underline{U}}))_{(6)}$

(2) $[3,2,4;2,1] - [4,2,3;2,1] + [4,2,2;3,1] - [4,1,2;3,2] - [3,2,2;4,1] + [3,1,2;4,2] - [2,1,2;4,3] = 0 \mod (F_6(\underline{\underline{U}}))_{(6)}$.

From these we eliminate: [3,1,4;2,1], [3,2,4;2,1].

(A.4.1.5) Table giving the images of the terms of v (p.63) under φ_i, $i \in \{1,2,\dots,9\}$ (p.64) in terms of basic commutators in 2 generators.

Images under $\varphi_1,\dots,\varphi_5$:

	φ_1	φ_2	φ_3	φ_4	φ_5
$e_{25}d_{25} \to$	$e_{25}d_{133}$	$e_{25}d_{128}$	x	x	x
$e_{26}d_{26} \to$	$e_{26}d_{134}$	0	0	0	0
$e_{27}d_{27} \to$	0	$e_{27}d_{129}$	x	x	x
$e_{28}d_{28} \to$	$e_{28}d_{135}$	0	$-e_{28}d_{128}$	x	x
$e_{29}d_{29} \to$	0	0	$-e_{29}d_{129}$	0	0
$e_{30}d_{30} \to$	0	$e_{30}d_{130}$	x	x	x
$e_{31}d_{31} \to$	$e_{31}d_{128}$	x	x	x	x
$e_{32}d_{32} \to$	$e_{32}d_{129}$	x	x	x	x
$e_{33}d_{33} \to$	0	$-e_{33}d_{132}+e_{33}d_{134}$	0	$e_{33}d_{20}$	$e_{33}d_{20}$
$e_{34}d_{34} \to$	$e_{34}d_{130}$	x	x	x	x
$e_{35}d_{35} \to$	0	0	$e_{35}d_{132}-e_{35}d_{134}-e_{35}d_{129}$	$e_{35}d_{20}$	$-e_{35}d_{20}$
$e_{36}d_{36} \to$	0	0	$-e_{36}d_{130}$	x	x
$e_{37}d_{37} \to$	0	0	0	0	0
$e_{38}d_{38} \to$	$e_{38}d_{133}$	0	0	0	0
$e_{39}d_{39} \to$	$e_{39}d_{134}$	0	0	0	0
$e_{40}d_{40} \to$	0	$e_{40}d_{132}$	0	0	0
$e_{41}d_{41} \to$	$e_{41}d_{135}$	0	0	0	0
$e_{42}d_{42} \to$	0	0	$-e_{42}d_{132}$	0	0

images under $\varphi_1,\ldots,\varphi_5$ continued

$e_{45}d_{43} \to$	0	$e_{43}d_{133}$	x	x	x
$e_{44}d_{44} \to$	$e_{44}d_{132}$	0	0	0	0
$e_{45}d_{45} \to$	0	$e_{45}d_{134}$	0	0	0
$e_{46}d_{46} \to$	0	0	0	$e_{46}d_{23}$	$e_{46}d_{23}$
$e_{47}d_{47} \to$	0	0	$e_{47}d_{132}$	$e_{47}d_{23}$	$-e_{47}d_{23}$
$e_{48}d_{48} \to$	0	$-e_{48}d_{135}$	x	x	x
$e_{49}d_{49} \to$	$-e_{49}d_{132}$	0	0	0	0
$e_{50}d_{50} \to$	0	0	$-e_{50}d_{133}$	x	x
$e_{51}d_{51} \to$	0	0	$-e_{51}d_{134}$	0	0
$e_{52}d_{52} \to$	0	$e_{52}d_{132}$	0	$e_{52}d_{23}$	$e_{52}d_{23}$
$e_{53}d_{53} \to$	0	0	$-e_{53}d_{134}$	$e_{53}d_{23}$	$-e_{53}d_{23}$
$e_{54}d_{54} \to$	0	0	$-e_{54}d_{135}$	x	x

Images under $\varphi_6, \ldots, \varphi_9$:

	φ_6	φ_7	φ_8	φ_9
$e_{25}d_{25} \to$	x	x	x	x
$e_{26}d_{26} \to$	$e_{26}d_{21}$	$-e_{26}d_{21}$	$e_{26}d_{23}+e_{26}d_{20}$	$e_{26}d_{23}+e_{26}d_{20}$
$e_{27}d_{27} \to$	x	x	x	x
$e_{28}d_{28} \to$	x	x	x	x
$e_{29}d_{29} \to$	$e_{29}d_{22}$	$e_{29}d_{22}$	$e_{29}d_{23}+e_{29}+d_{20}$	$-e_{29}d_{23}-e_{29}d_{20}$
$e_{30}d_{30} \to$	x	x	x	x
$e_{31}d_{31} \to$	x	x	x	x
$e_{32}d_{32} \to$	x	x	x	x
$e_{33}d_{33} \to$	$e_{33}d_{21}$	$e_{33}d_{21}$	0	0
$e_{34}d_{34} \to$	x	x	x	x
$e_{35}d_{35} \to$	$e_{35}d_{22}$	$e_{35}d_{22}$	0	0
$e_{36}d_{36} \to$	x	x	x	x
$e_{37}d_{37} \to$		$-e_{37}d_{24}$	$e_{37}d_{23}$	$e_{37}d_{23}$
$e_{38}d_{38} \to$		$-e_{38}d_{23}$	0	0
$e_{39}d_{39} \to$		$-e_{39}d_{24}$	0	0
$e_{40}d_{40} \to$	$d_{40}d_{24}$	$e_{40}d_{24}$	0	0
$e_{41}d_{41} \to$	0	0	$-e_{41}d_{24}$	$e_{41}d_{24}$
$e_{42}d_{42} \to$	0	0	$-e_{42}d_{23}$	$e_{42}d_{23}$
$e_{43}d_{43} \to$	x	x	x	x
$e_{44}d_{44} \to$	$e_{44}d_{24}$	$-e_{44}d_{24}$	0	0

images under $\varphi_6,\ldots,\varphi_9$ continued:

$e_{45}d_{45} \to$	$e_{45}d_{24}$	$e_{45}d_{24}$	0	0
$e_{46}d_{46} \to$	$e_{46}d_{24}$	$e_{46}d_{24}$	0	0
$e_{47}d_{47} \to$	0	0	0	0
$e_{48}d_{48} \to$	x	x	x	x
$e_{49}d_{49} \to$	0	0	$-e_{49}d_{23}$	$-e_{49}d_{23}$
$e_{50}d_{50} \to$	x	x	x	x
$e_{51}d_{51} \to$	0	0	$-e_{51}d_{23}$	$e_{51}d_{23}$
$e_{52}d_{52} \to$	0	0	0	0
$e_{53}d_{53} \to$	0	0	0	0
$d_{54}d_{54} \to$	x	x	0	0 .

Some equations (in coefficients of basic commutators of type (x,y,z,t,u)) obtained from the column headed by

(a) φ_1 : (1) $e_{31} = 0$,

(2) $e_{32} = 0$,

(3) $e_{34} = 0$,

(b) φ_2 : (4) $e_{25} = 0$,

(5) $e_{27} = 0$,

(6) $e_{30} = 0$,

(c) φ_3 : (7) $e_{28} = 0$,

(8) $e_{29} = 0$,

(9) $e_{36} = 0$,

(d) φ_4 : (10) $e_{33} + e_{35} = 0$,

(e) φ_5 : (11) $e_{33} - e_{35} = 0$,

(f) φ_6 : (12) $e_{26} + e_{29} = 0$,

(g) φ_7 : (13) $-e_{26} + e_{29} = 0$.

Equations (10), (11) imply that $e_{33} = 0$, $e_{35} = 0$, and equations (12), (13) imply that $e_{26} = 0$, $e_{29} = 0$. Hence the coefficients of basic commutators of type (x,y,z,t,u) in the left-most column are all zero. Equations (in coefficients of basic commutators of type $(x,y,z;t,u)$ now of course) obtained from the column headed by

(h) φ_1 : (14) $e_{38} = 0$,

(15) $e_{39} = 0$,

(16) $e_{41} = 0$,

(17) $e_{44} - e_{49} = 0$,

(i) φ_2 : (18) $e_{43} = 0$,

(19) $e_{45} = 0$,

(20) $e_{48} = 0$,

(21) $e_{40} + e_{52} = 0$,

(j) φ_3 : (22) $e_{50} = 0$,

(23) $e_{54} = 0$,

(24) $-e_{42} + e_{47} = 0$,

(25) $e_{51} + e_{53} = 0$,

(k) φ_4 : (26) $e_{46} + e_{47} + e_{52} + e_{53} = 0,$

(l) φ_5 : (27) $e_{46} - e_{47} + e_{52} - e_{53} = 0,$

(m) φ_6 : (28) $e_{37} + e_{39} + e_{40} + e_{44} + e_{45} + e_{46} = 0,$

(n) φ_7 : (29) $-e_{37} - e_{39} + e_{40} - e_{44} + e_{45} + e_{46} = 0,$

(o) φ_8 : (30) $e_{37} - e_{42} - e_{49} - e_{51} = 0,$

(p) φ_9 : (31) $e_{37} + e_{42} - e_{49} + e_{51} - 0.$

From equations (26), (27); (15), (19), (28), (29); (30), (31), we deduce:

(32) $e_{46} + e_{52} = 0,$

(33) $e_{47} + e_{53} = 0,$

(34) $e_{40} + e_{46} = 0,$

(35) $e_{37} + e_{44} = 0,$

(36) $e_{37} - e_{49} = 0,$

(37) $e_{42} + e_{51} = 0\ .$

Now (21) - (32), (35) - (17), (33) - (25) gives respectively:

(38) $e_{40} - e_{46} = 0,$

(39) $e_{37} + e_{49} = 0,$

(40) $e_{47} - e_{51} = 0,$

and (40) - (24) gives:

(41) $e_{42} - e_{51} = 0,$

so that from (34), (38); (36), (39); (37), (43), we at once obtain that $e_{37}, e_{40}, e_{46}, e_{47}, e_{49}, e_{51}$ are trivial. Equations (17), (21), (24), (25) now reduce respectively to $e_{44} = 0$, $e_{52} = 0$, $e_{42} = 0$, $e_{53} = 0$.

Hence in v the commutators of weight 5 in precisely the generators 1,2,3 have trivial coefficients.

(A.4.1.6) <u>To show that</u> $v_5^{(1234')}$ (p.61) <u>is trivial</u>.

We have:

$$\begin{aligned} v_5^{(1234')} &= e_{72}[4,3,4;2,1] + e_{78}[4,2,4;3,1] + e_{84}[4,1,4;3,2] \\ &+ e_{89}[3,2,4;4,1] + e_{90}[4,2,3;4,1] + [e_{95}[3,1,4;4,2] \\ &+ e_{96}[4,1,3;4,2] + e_{100}[2,1,4;4,3] + e_{102}[4,1,2;4,3]. \end{aligned}$$

Hence,

$$\begin{aligned} x = {} & e_{72}[4,3,5;2,1] + e_{72}[5,3,4;2,1] + e_{78}[4,2,5;3,1] + e_{78}[5,2,4;3,1] \\ &+ e_{84}[4,1,5;3,2] + e_{84}[5,1,4;3,2] + e_{89}[3,2,5;4,1] + e_{89}[3,2,4;5,1] \\ &+ e_{90}[5,2,3;4,1] + e_{90}[4,2,3;5,1] + e_{95}[3,1,5;4,2] + e_{95}[3,1,4;5,2] \\ &+ e_{96}[5,1,3;4,2] + e_{96}[4,1,3;5,2] + e_{100}[2,1,5;4,3] + e_{100}[2,1,4,5,3] \\ &+ e_{102}[5,1,2;4,3] + e_{102}[4,1,2;5,3]. \end{aligned}$$

Now relations (r.1), (r.2) mod weight 6 give

$$\begin{aligned} e_{100}[2,1,5;4,3] &= e_{100}\{-[4,3,5;2,1]+[4,2,5;3,1]-[4,1,5;3,2]-[3,2,5;4,1]+ \\ &\qquad [3,1,5;4,2]\}. \\ e_{100}[2,1,4;5,3] &= e_{100}\{-[5,3,4;2,1]+[5,2,4;3,1]-[5,1,4;3,2]-[3,2,4;5,1]+ \\ &\qquad [3,1,4;5,2]\}. \end{aligned}$$

Substituting these into the expression for x, arranging like - terms together, and then applying Lemma (4.3), we obtain the following equations in the coefficients of the commutators in $v_5^{(1234')}$.

(1) $e_{90} = 0$,

(2) $e_{96} = 0$,

(3) $e_{102} = 0$,

(4) $e_{95} + e_{100} = 0$,

(5) $e_{89} - e_{100} = 0$,

(6) $e_{84} - e_{100} = 0$,

(7) $e_{78} + e_{100} = 0$,

(8) $e_{72} - e_{100} = 0$.

Thus $v_5^{(1234')}$ can be written in the form:

$$v_5^{(1234')} = e_{100}\{[4,3,4;2,1]-[4,2,4;3,1]+[4,1,4;3,2]+[3,2,4;4,1] -[3,1,4;4,2]+[2,1,4;4,3]\}.$$

But this is trivial in $F_6(\underline{\underline{U}})$, since the expression inside $\{\ \}$ is obtained from $w_7(2,6)$ by the substitution $x_5 \to 4$ and $x_i \to i$ for all $i \in \{1,2,3,4\}$. Hence, $v_5^{(1234')}$ is trivial.

(A.4.1.7) Table giving the images of the terms of $v_5^{(1'234)}$ (p. 67) in terms of basic commutators in 3 generators.

	φ_{21}	φ_{22}	φ_{23}
$e_{69}[4,1,3;2,1] \to$	$e_{69}[2,1,3;2,1]$	$e_{69}[3,1,3;2,1]$	x
$e_{73}[2,1,4;3,1] \to$	$e_{73}[2,1,2;3,1]$	$e_{73}[2,1,3;3,1]$	$e_{73}[2,1,1;3,1]$
$e_{75}[4,1,2;3,1] \to$	$e_{75}[2,1,2;3,1]$	$e_{75}[3,1,2;3,1]$	x
$e_{81}[4,1,1;3,2] \to$	$e_{81}[2,1,1;3,2]$	$e_{81}[3,1,1;3,2]$	0
$e_{85}[2,1,3;4,1] \to$	$e_{85}[2,1,3;2,1]$	$e_{85}[2,1,3;3,1]$	0
$e_{86}[3,1,2;4,1] \to$	$e_{86}[3,1,2;2,1]$	x	x
$e_{92}[3,1,1;4,2] \to$	e_{92} 0	$e_{92}[3,1,1;3,2]$	$-e_{92}[3,1,1;2,1]$
$e_{97}[2,1,1;4,3] \to$	$e_{97}-[2,1,1;3,2]$	0	$-e_{97}[2,1,1;3,1]$

Equations obtained from column headed by

(a) φ_{21} : (1) $e_{86} = 0$,

(2) $e_{69} + e_{85} = 0$,

(3) $e_{73} + e_{75} = 0$,

(4) $e_{81} - e_{97} = 0$.

(b) φ_{22} : (5) $e_{69} = 0$,

(6) $e_{75} = 0$,

(7) $e_{73} + e_{85} = 0$,

(8) $e_{81} + e_{92} = 0$.

(c) φ_{23} : (9) $e_{92} = 0$,

(10) $e_{73} - e_{97} = 0$.

Equations (1), (5), (6), (2), (3), (9), (8), (4) give immediately that the coefficients e_{69}, e_{73}, e_{75}, e_{81}, e_{85}, e_{86}, e_{92}, e_{97} are trivial.

(A.4.1.8) <u>Relations obtained from</u> $w_1(2,6) = [x_4,x_3,x_5;x_2,x_1,x_6] + [x_4,x_3,x_6;x_2,x_1,x_5]$ (p.57) <u>and elimination of commutators from them.</u>

Now to each basic commutator of type (3,3) in $F_6(\underline{\underline{U}})$ there corresponds a relation in basic commutators of the same type. For example, corresponding to the basic commutator [5,3,6;2,1,4] there is the relation [5,3,6;2,1,4] + [5,3,4;2,1,6] = 0. Thus we obtain the following relations.

(1) [5,3,6;2,1,4] = -[5,3,4;2,1,6],

(2) [6,3,5;2,1,4] = -[6,3,4;2,1,5],

(3) [4,3,6;2,1,5] = -[4,3,5;2,1,6],

(4) [5,2,6;3,1,4] = -[5,2,4;3,1,6],

(5) [6,2,5;3,1,4] = -[6,2,4;3,1,5],

(6) [4,2,6;3,1,5] = -[4,2,5;3,1,6],

(7) [5,1,6;3,2,4] = -[5,1,4;3,2,6],

(8) [6,1,5;3,2,4] = -[6,1,4;3,2,5],

(9) [4,1,6;3,2,5] = -[4,1,5;3,2,6],

(10) [5,2,6;4,1,3] = -[5,2,3;4,1,6],

(11) [6,2,5;4,1,3] = -[6,2,3;4,1,5],

(12) [5,1,6;4,2,3] = -[5,1,3;4,2,6],

(13) [6,1,5;4,2,3] = -[6,1,3;4,2,5],

(14) $[6,2,4;5,1,3] = -[6,2,3;5,1,4]$,

(15) $[6,1,4;5,2,3] = -[6,1,3;5,2,4]$,

(16) $[5,4,6;2,1,3] = [4,3,5;2,1,6] - [5,3,4;2,1,6]$ or $d_{490}=d_{496}-d_{497}$,

(17) $[6,4,5;2,1,3] = [4,3,6;2,1,5] - [6,3,4;2,1,5]$ or $d_{491}=d_{494}-d_{495}$,

(18) $[5,4,6;3,1,2] = [4,2,5;3,1,6] - [5,2,4;3,1,6]$ or $d_{498}=d_{504}-d_{505}$,

(19) $[6,4,5;3,1,2] = [4,2,6;3,1,5] - [6,2,4;3,1,5]$ or $d_{499}=d_{502}-d_{503}$,

(20) $[5,3,6;4,1,2] = -[4,1,6;3,2,5] - [5,2,3;4,1,6]$ or $d_{512}=-d_{508}-d_{517}$,

(21) $[6,3,5;4,1,2] = -[4,1,5;3,2,6] - [6,2,3;4,1,5]$ or $d_{513}=-d_{510}-d_{516}$,

(22) $[6,1,2;4,3,5] = [6,1,5;3,2,4] - [6,1,5;4,2,3]$ or $d_{522}=d_{507}-d_{519}$,

(23) $[5,1,2;4,3,6] = [5,1,6;3,2,4] - [5,1,6;4,2,3]$ or $d_{523}=d_{506}-d_{518}$,

(24) $[6,3,4;5,1,2] = -[5,1,4;3,2,6] - [6,2,3;5,1,4]$ or $d_{524}=-d_{511}-d_{526}$,

(25) $[6,1,2;5,3,4] = [6,1,4;3,2,5] - [6,1,4;5,2,3]$ or $d_{529}=d_{509}-d_{527}$.

From relations (1) - (15). The following commutators (they are the right-hand members of the relations) are eliminated.

d_{495}, d_{496}, d_{497}, d_{503}, d_{504}, d_{505}, d_{509}, d_{510}, d_{511},
d_{516}, d_{517}, d_{520}, d_{521}, d_{526}, d_{528}.

Equations (16) - (25) now reduce to:

(26) $d_{494} = -d_{490} + d_{492}$

(27) $d_{494} = d_{491} - d_{493}$

(28) $d_{502} = -d_{498} + d_{500}$

(29) $d_{502} = d_{499} - d_{501}$

(30) $d_{514} = d_{508} + d_{512}$

(31) $d_{515} = -d_{508} + d_{513}$

(32) $d_{522} = d_{507} - d_{519}$

(33) $d_{523} = d_{506} - d_{518}$

(34) $d_{525} = -d_{506} + d_{524}$

(35) $d_{529} = -d_{507} - d_{527}$.

We use relations (26), (28), (30), (31), (32), (33), (34), (35) to eliminate the commutators (they are the extreme left-hand members of the relations):

$$d_{494},\ d_{502},\ d_{514},\ d_{515},\ d_{522},\ d_{523},\ d_{525},\ d_{529},$$

and relations (27), (29) then reduce to:

(36) $d_{493} = d_{490} + d_{491} - d_{492}$,

(37) $d_{501} = d_{498} + d_{499} - d_{500}$,

from which we eliminate further the commutators:

$$d_{493},\ d_{501}.$$

(A.4.1.9) <u>Relations obtained from</u> $w_3(2,6) = [x_4,x_3,x_6;x_2,x_1,x_5] - [x_4,x_2,x_6;x_3,x_1,x_5] + [x_4,x_1,x_6;x_3,x_2,x_5]$ (p. 57) <u>and elimination of basic commutators from them.</u>

In (A.4.1.8), we eliminated 25 of the 40 basic commutators of type (3,3) in precisely 6 generators. Now to every commutator of type (3,3) say $[a,b,c;d,e,f]$ in $F_6(\underline{\underline{U}})$, there corresponds a relation given by the substitution: $x_1 \to b$, $x_2 \to e$, $x_3 \to d$, $x_4 \to a$, $x_5 \to f$, $x_6 \to c$.

Thus corresponding to the remaining 15 basic commutators of type (3,3) in precisely 6 generators there are the following distinct relations.

(1) $[5,4,6;2,1,3] - [5,2,6;4,1,3] + [5,1,6;4,2,3] = 0$, or $d_{490}-d_{514}+d_{518} = 0$,

(2) $[6,4,5;2,1,3] - [6,2,5;4,1,3] + [6,1,5;4,2,3] = 0$, or $d_{491}-d_{515}+d_{519} = 0$,

(3) $[5,3,6;2,1,4] - [5,2,6;3,1,4] + [5,1,6;3,2,4] = 0$, or $d_{492}-d_{500}+d_{506} = 0$,

(4) $[5,4,6;3,1,2] - [5,1,6;3,2,4] - [5,3,6;4,1,2] + [5,1,6;4,2,3] = 0$, or

$$d_{498}-d_{506}-d_{512}+d_{518} = 0,$$

(5) $[6,4,5;3,1,2] - [6,1,5;3,2,4] - [6,3,5;4,1,2] + [6,1,5;4,2,3] = 0$,

$$d_{499}-d_{507}-d_{513}+d_{519} = 0, \quad \text{or},$$

(6) $[6,3,5;2,1,4] - [6,2,5;3,1,4] + [6,1,5;3,2,4] = 0$, or

$$d_{493}-d_{501}+d_{507} = 0,$$

(7) $[4,3,6;2,1,5] - [4,2,6;3,1,5] + [4,1,6;3,2,5] = 0$, or

$$d_{494}-d_{502}+d_{508} = 0,$$

(8) $[5,4,6;3,1,2] - [6,4,5;3,1,2] + [6,1,4;3,2,5] + [6,3,4;5,1,2] -$

$[6,1,4;5,2,3] = 0$, or $d_{498}-d_{499}+d_{509}+d_{524}-d_{527} = 0$,

(9) $-[5,4,6;2,1,3] + [6,4,5;2,1,3] - [6,2,4;5,1,3] + [6,1,4;5,2,3] = 0$ or

$$-d_{490}+d_{491}-d_{525}+d_{527} = 0.$$

If we eliminate from these relations the basic commutators that are eliminated in (A.4.1.8), they reduce to:

(10) $d_{490} - d_{508} - d_{512} + d_{518} = 0$,

(11) $d_{491} + d_{508} - d_{513} + d_{519} = 0$,

(12) $d_{492} - d_{500} + d_{506} = 0$,

(13) $d_{498} - d_{506} - d_{512} + d_{518} = 0$,

(14) $d_{499} - d_{507} - d_{513} + d_{519} = 0$,

(15) $d_{490} + d_{491} - d_{492} - d_{498} - d_{499} + d_{500} + d_{507} = 0$,

(16) $-d_{490} + d_{492} + d_{498} - d_{500} + d_{508} = 0$,

(17) $d_{498} - d_{499} - d_{507} + d_{524} - d_{527} = 0$,

(18) $-d_{490} + d_{491} + d_{506} - d_{524} - d_{527} = 0$,

and we eliminate:

$d_{506}, d_{507}, d_{508}, d_{518}, d_{519}, d_{527}$.

(A.4.1.10) Relations obtained from $w_6(2,6) = -[x_6,x_4,x_5;x_2,x_1,x_3] + [x_5,x_3,x_6;x_2,x_1,x_4]$ (p. 57) and elimination of basic commutators from them.

By permuting the generators we obtain the following relations from $w_6(2,6)$.

(1) $-[6,4,5;2,1,3] + [5,3,6;2,1,4] = 0$, or $-d_{491}+d_{492} = 0$,

(2) $[6,4,5;3,1,2] - [5,2,6;3,1,4] = 0$, or $d_{499}-d_{500} = 0$,

(3) $-[5,4,6;2,1,3] - [6,4,5;2,1,3] + 2[5,3,6;2,1,4] + [5,4,6;3,1,2]$

$-[5,2,6;3,1,4] - [5,3,6;4,1,2] + [6,3,5;4,1,2] = 0$, or

$-d_{490} - d_{491} + 2d_{492} + d_{498} - d_{500} - d_{512} + d_{513} = 0$,

(4) $-[6,4,5;2,1,3] + 2[5,3,6;2,1,4] + [5,4,6;3,1,2] - [5,2,6;3,1,4] -$

$[5,3,6;4,1,2] + [6,3,4;5,1,2] = 0$, or $-d_{491}+d_{492}+d_{498}-d_{500}-d_{512}+d_{524} = 0$.

From these we eliminate:

d_{492}, d_{500}, d_{513}, d_{524}.

(A.4.1.11) Relations obtained from $w_2(2,6) = [x_2,x_1,x_5,x_6;x_4,x_3] - [x_4,x_3,x_5,x_6;x_2,x_1]$ (p.57) in $F_6(\underline{\underline{U}})$ and elimination of basic commutators from them.

To each basic commutator of type (4,2) in $F_6(\underline{\underline{U}})$ there corresponds a relation in basic commutator of the same type. Thus corresponding to such basic commutators in precisely 6 generators in $F_6(\underline{\underline{U}})$, we have the following relations.

(1) $[3,2,5,6;4,1] = [4,1,5,6;3,2]$

(2) $[3,1,5,6;4,2] = [4,2,5,6;3,1]$

(3) $[2,1,5,6;4,3] = [4,3,5,6;2,1]$

(4) $[3,2,4,6;5,1] = [5,1,4,6;3,2]$

(5) $[4,2,3,6;5,1] = [5,1,3,6;4,2]$

(6) $[3,1,4,6;5,2] = [5,2,4,6;3,1]$

(7) $[4,1,3,6;5,2] = [5,2,3,6;4,1]$

(8) $[2,1,4,6;5,3] = [5,3,4,6;2,1]$

(9) $[3,2,4,5;6,1] = [6,1,4,5;3,2]$

(10) $[4,2,3,5;6,1] = [6,1,3,5;4,2]$

(11) [5,2,3,4;6,1] = [6,1,3,4;5,2]

(12) [3,1,4,5;6,2] = [6,2,4,5;3,1]

(13) [4,1,3,5;6,2] = [6,2,3,5;4,1]

(14) [5,1,3,4;6,2] = [6,2,3,4;5,1]

(15) [2,1,4,5;6,3] = [6,3,4,5;2,1]

(16) [5,1,2,6;4,3] = -[3,2,4,6;5,1] + [4,2,3,6;5,1], or $d_{546} = -d_{548}+d_{549}$,

(17) [6,1,2,5;4,3] = -[3,2,4,5;6,1] + [4,2,3,5;6,1], or $d_{547} = -d_{560}+d_{561}$,

(18) [4,1,2,6;5,3] = -[3,2,5,6;4,1] + [5,2,3,6;4,1], or $d_{555} = -d_{589}+d_{540}$,

(19) [6,1,2,4;5,3] = -[3,2,4,5;6,1] + [5,2,3,4;6,1], or $d_{556} = -d_{560}+d_{562}$,

(20) [2,1,3,6;5,4] = -[4,3,5,6;2,1] + [5,3,4,6;2,1], or $d_{557} = -d_{530}+d_{531}$,

(21) [3,1,2,6;5,4] = -[4,2,5,6;3,1] + [5,2,4,6;3,1], or $d_{558} = -d_{533}+d_{534}$,

(22) [6,1,2,3;5,4] = -[4,2,3,5;6,1] + [5,2,3,4;6,1], or $d_{559} = -d_{561}+d_{562}$,

(23) [4,1,2,5;6,3] = -[3,2,5,6;4,1] + [6,2,3,5;4,1], or $d_{567} = -d_{539}+d_{541}$,

(24) [5,1,2,4;6,3] = -[3,2,4,6;5,1] + [6,2,3,4;5,1], or $d_{568} = -d_{548}+d_{550}$,

(25) [2,1,3,5;6,4] = -[4,3,5,6;2,1] + [6,3,4,5;2,1], or $d_{569} = -d_{530}+d_{532}$,

(26) [3,1,2,5;6,4] = -[4,2,5,6;3,1] + [6,2,4,5;3,1], or $d_{570} = -d_{533}+d_{535}$,

(27) [5,1,2,3;6,4] = -[4,2,3,6;5,1] + [6,2,3,4;5,1], or $d_{571} = -d_{549}+d_{550}$,

(28) [2,1,3,4;6,5] = -[5,3,4,6;2,1] + [6,3,4,5;2,1], or $d_{572} = -d_{531}+d_{532}$,

(29) [3,1,2,4;6,5] = -[5,2,4,6;3,1] + [6,2,4,5;3,1], or $d_{573} = -d_{534}+d_{535}$,

(30) [4,1,2,3;6,5] = -[5,2,3,6;4,1] + [6,2,3,5;4,1], or $d_{574} = -d_{540}+d_{541}$.

From relations (1) to (15), the following commutators (they are the left-hand members of the relations) are eliminated.

d_{539}, d_{542}, d_{545}, d_{548}, d_{549}, d_{551}, d_{552}, d_{554}, d_{560},
d_{561}, d_{562}, d_{563}, d_{564}, d_{565}, d_{566}.

Relations (16) - (30) then reduce to:

(31) $d_{546} = -d_{537} + d_{543}$,

(32) $d_{547} = -d_{538} + d_{544}$,

(33) $d_{555} = -d_{536} + d_{540}$,

(34) $d_{556} = -d_{538} + d_{553}$,

(35) $d_{557} = -d_{530} + d_{531}$,

(36) $d_{558} = -d_{533} + d_{534}$,

(37) $d_{559} = -d_{544} + d_{553}$,

(38) $d_{567} = -d_{536} + d_{541}$,

(39) $d_{568} = -d_{537} + d_{550}$,

(40) $d_{569} = -d_{530} + d_{532}$,

(41) $d_{570} = -d_{533} + d_{535}$,

(42) $d_{571} = -d_{543} + d_{550}$,

(43) $d_{572} = -d_{531} + d_{532}$,

(44) $d_{573} = -d_{534} + d_{535}$,

(45) $d_{574} = -d_{540} + d_{541}$,

from which we eliminate:

d_{546}, d_{547}, d_{555}, d_{556}, d_{557}, d_{558}, d_{559}, d_{567}, d_{568}, d_{569}, d_{570}, d_{571}, d_{572}, d_{573}, d_{574}.

(A.4.1.12) Relations obtained from $w_4(2,6) = -[x_3,x_2,x_5,x_6;x_4,x_1] + [x_3,x_1,x_5,x_6;x_4,x_2] - [x_2,x_1,x_5,x_6;x_4,x_3]$ (p.51) and elimination of basic commutators from them.

In (A.4.1.11) we eliminated 30 of the 45 basic commutator of type (4,2). As in (A.4.1.9), we obtain from $w_4(2,6)$ the distinct relations:

(1) $[4,3,5,6;2,1] - [3,1,5,6;4,2] + [4,1,5,6;3,2] = 0$, or $d_{530}-d_{542}+d_{536} = 0$,

(2) $[5,3,4,6;2,1] - [3,1,4,6;5,2] + [5,1,4,6;3,2] = 0$, or $d_{531}-d_{551}+d_{537} = 0$,

(3) $[6,3,4,5;2,1] - [3,1,4,5;6,2] + [6,1,4,5;3,2] = 0$, or $d_{532}-d_{563}+d_{538} = 0$,

(4) $-[5,2,3,6;4,1] + [5,1,3,6;4,2] + [2,1,3,6;5,4] = 0$, or $-d_{540}+d_{543}+d_{557} = 0$,

(5) $-[6,2,3,5;4,1] + [6,1,3,5;4,2] + [2,1,3,5;6,4] = 0$, or $-d_{541}+d_{544}+d_{569} = 0$,

(6) $-[6,2,3,4;5,1] + [6,1,3,4;5,2] + [2,1,3,4;6,5] = 0$, or $-d_{550}+d_{553}+d_{572} = 0$.

If we eliminate from these relations the basic commutators that are eliminated in (A.4.1.4), they reduce to:

(7) $d_{530} - d_{533} + d_{536} = 0$,

(8) $d_{531} - d_{534} + d_{537} = 0$,

(9) $d_{532} - d_{535} + d_{538} = 0$,

(10) $-d_{530} + d_{531} - d_{540} + d_{543} = 0$,

(11) $-d_{530} + d_{532} - d_{541} + d_{544} = 0$,

(12) $-d_{531} + d_{532} - d_{550} + d_{553} = 0$,

and we eliminate from them :

$d_{536}, d_{537}, d_{538}, d_{543}, d_{544}, d_{553}$.

(A.4.1.13) Table of images of the terms of v_2 (p.70) under φ_{3j}, $j \in \{1,2,\dots,6\}$ (p. 70) in terms of basic commutators in 2 generators.

	φ_{31}	φ_{32}	φ_{33}	φ_{34}	φ_{35}	φ_{36}
$e_{485}d_{485} \rightarrow$	0	0	0	0	$e_{485}d_{132}$	0
$e_{486}d_{486} \rightarrow$	0	0	0	0	$e_{486}d_{132}$	$-e_{486}d_{132}$
$e_{493}d_{493} \rightarrow$	0	$e_{493}d_{132}$	$-e_{493}d_{132}$	x	x	x
$e_{494}d_{494} \rightarrow$	0	$e_{494}d_{132}$	0	$-e_{494}d_{132}$	x	x
$e_{507}d_{507} \rightarrow$	$e_{507}d_{132}$	x	x	x	x	x

Equations obtained from the column headed by

(a) φ_{31} : (1) $e_{507} = 0$,

(b) φ_{32} : (2) $e_{493} + e_{494} = 0$,

(c) φ_{33} : (3) $e_{493} = 0$,

(d) φ_{34} : (4) $e_{494} = 0$,

(e) φ_{35} : (5) $e_{485} + e_{486} = 0$,

(f) φ_{36} : (6) $e_{486} = 0$.

We see immediately that the coefficients e_{485}, e_{486}, e_{493}, e_{494}, e_{507} are trivial.